$(2P_2, K_4)$-Free Graphs are 4-Colorable

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Outline

Graph Coloring

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Given a graph $G = (V, E)$, a function $\phi : V(G) \rightarrow \{1, 2, \ldots, k\}$ is a *k-coloring* of G if $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$.

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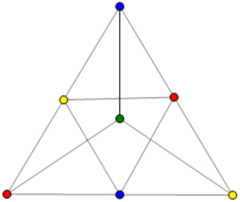

Figure: $\chi(H) = 4$.

Clique Number

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No! Myceilsky Construction.

χ-Bounded

Special Graph Classes.

A class $\mathcal{G}$ of graphs is χ-bounded if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{G}$. The function f is called a χ-bounding function for $\mathcal{G}$.

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Example

- Bipartite graphs: $\chi = \omega = 2. \rightarrow f(x) = x$.
- Berge graphs: $\chi = \omega$. (Strong Perfect Graph Theorem.)

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Known

- paths (Gyárfás 1987)
- trees of radius at most 2 (Kierstead and Penrise 1994)
- special trees of radius at most 3 (Kierstead and Zhu 2004)
- Two-legged caterpillars and double-headed brooms (Chudnovsky, Scott and Seymour 2017)

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We are interested in $2P_2$-free graphs.

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Let G be a $2P_2$-free graph. Then $\chi \leq \binom{\omega+1}{2}$.

Proof. Choose a maximum clique K and show that G can be partitioned into $\binom{\omega}{2} + \omega = \binom{\omega+1}{2}$ stable sets.

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Questions by Wagon

- What is the optimal χ-bounding function?
- What is the optimal bound when $\omega = 3$?

The Main Result

If $\omega = 3$, we know that

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- The bound is optimal: W_5 and $\overline{C_7}$.
- The bound is a Vizing-type bound.
- Our proof is algorithmic.

An Application

- Determining $\chi(G)$ for a graph G is NP-complete.
- Numerous studies on H-free graphs.

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Three Open Problems for (H_1, H_2)-Free Graphs

- $(4P_1, C_4)$.
- $(4P_1, K_{1,3})$.
- $(4P_1, P_2 + 2P_1, K_{1,3})$.

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Let G be a $(4P_1, C_4)$-free graph. Then $\overline{G}$ is $(2P_2, K_4)$-free.

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- Since G is C_4-free, both $G[K_1 \cup K_2]$ and $G[K_3 \cup K_4]$ are chordal.
- The value $\chi(G[K_1 \cup K_2]) + \chi(G[K_3 \cup K_4])$ provides a 2-approximation.

Proof Sketch

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High level idea:

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Every $(2P_2, K_4)$-free graph is 4-colorable.

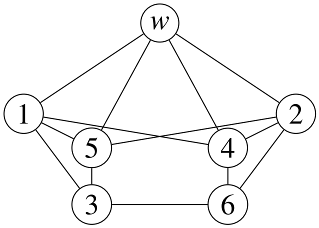
High level idea:

- Reduce to C_5-free graphs and then use a theorem of Chudnovsky, Robertson, Seymour and Thomas.

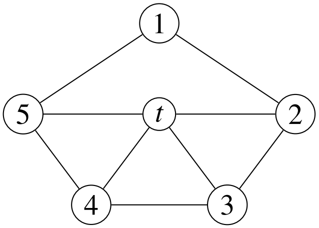
M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas.

K_4-free graphs with no odd holes.
J. Combin. Theory, Ser. B, 100:313–331, 2010.

Difficulty



H_1



H_2

Need to do this in a series of four steps.

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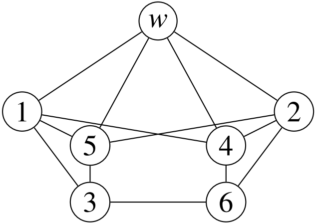

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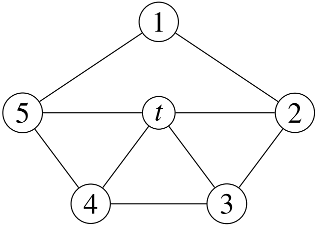

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- If G contains an H_1, the theorem holds.

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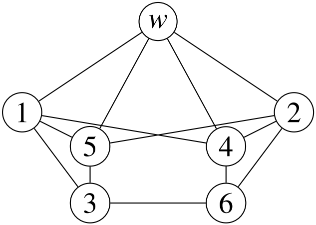

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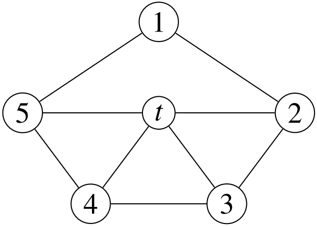

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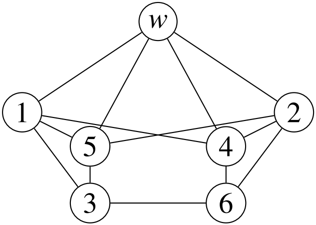
- If G contains an H_1, the theorem holds.
- If G contains an H_2, the theorem holds.

Difficulty



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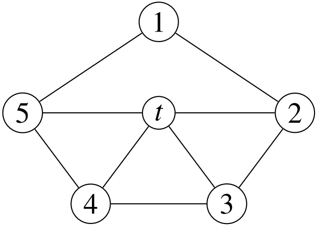

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- If G contains a W_5, the theorem holds.

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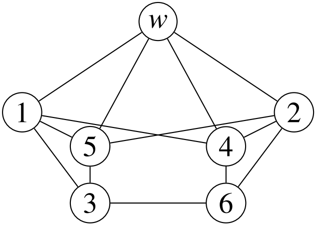

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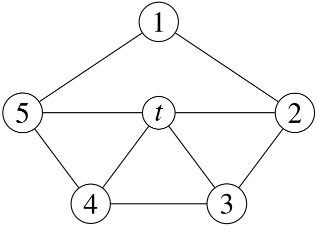

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- If G contains an H_1, the theorem holds.
- If G contains an H_2, the theorem holds.
- If G contains a W_5, the theorem holds.
- If G contains a C_5, the theorem holds.

Future Research

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Open

- What is the optimal χ-bounding function for $2P_2$-free graphs? $\omega^{1+\epsilon}$?
- What is the optimal χ-bounding function when $\omega = 4$? $\rightarrow 5 \leq \chi \leq 10.$