Critical permutation sets for generalized of signed graph coloring

Hao Qi[1] **Tsai-Lien Wong**[2] **Xuding Zhu**[3]

[1]Academia Sinica
[2]National Sun Yat-sen University
[3]Zhejiang Normal University

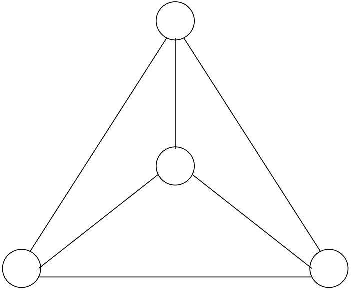
August. 20 / Academia Sinica

Outline

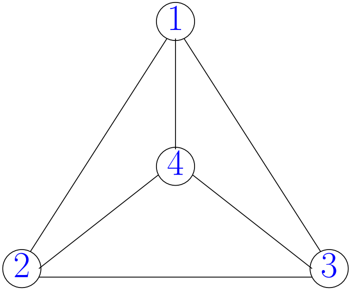
Outline

coloring

A *k-coloring* of a graph G is a mapping $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \varphi(v)$ for any edge $e = uv \in E(G)$.



A *k-coloring* of a graph G is a mapping $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \varphi(v)$ for any edge $e = uv \in E(G)$.



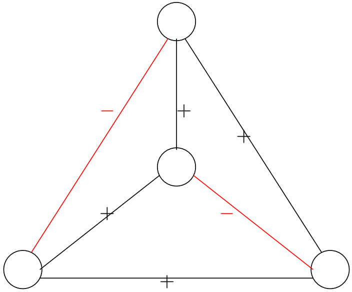
A *k-list coloring* of G is a k-coloring φ such that $\varphi(v) \in L(v)$ for any $v \in V(G)$, where L is a mapping $L : V(G) \to \mathbb{N}$ and $|L(v)| = k$.

Outline

1. Introduction
   - Coloring
   - Signed coloring
2. Critical permutation set
   - Generalized signed coloring
   - S is Critical
3. Results
   - $S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical
   - $\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_q$ is critical
4. Summary

Signed graph

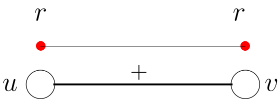
A *signed graph* is a pair (G, σ), where G is a graph and $\sigma : E(G) \to \{1, -1\}$ is a mapping which assigns to each edge e a sign σ_e.

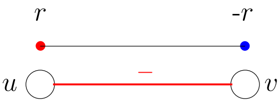


MRS-Z-k-coloring

For an integer k, let $N_k = \begin{cases} \{0, \pm 1, \pm 2, \ldots, \pm r\}, & \text{if } k = 2r+1, \\ \{\pm 1, \pm 2, \ldots, \pm r\}, & \text{if } k = 2r. \end{cases}$

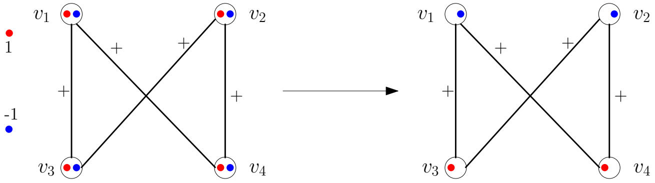
A *MRS-Z-k-coloring* of a signed graph (G, σ) is a mapping $\varphi : V(G) \to N_k$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

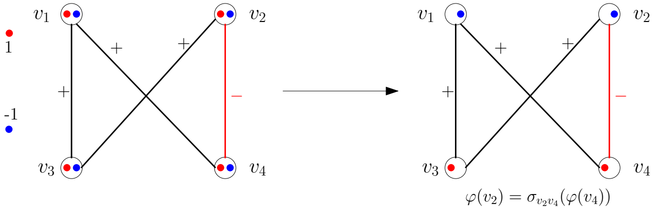




T. Zaslavsky. Signed graph coloring. *Discrete Math.*,39(2): 215-228, 1982.
E. Máčajová, A. Raspaud, M. Škoviera.The chromatic number of a signed graph. *Electron. J. Combin.* 23 (1) (2016) #P1.14.

MRS-Z-k-coloring





KS-k-coloring

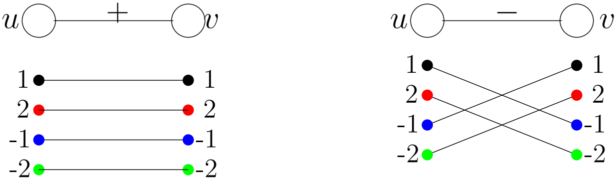
A *KS-k-coloring* of (G,σ) is a mapping $\varphi : V \to \mathbb{Z}_k$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

✠ Y. Kang and E. Steffen. The chromatic spectrum of signed graphs. *Discrete Math.*, 339: 2660–2663, 2016.

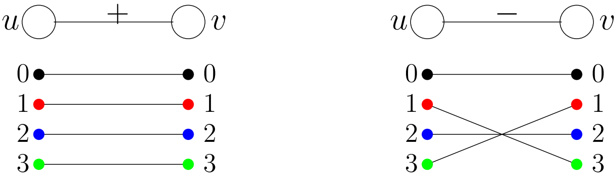
✠ Y. Kang and E. Steffen. Circular coloring of signed graphs. *J. Graph Theory*, 87(2): 135–148, 2018.

MRS-Z-k-coloring V.S. KS-k-coloring

MRS-Z-4-coloring



KS-4-coloring

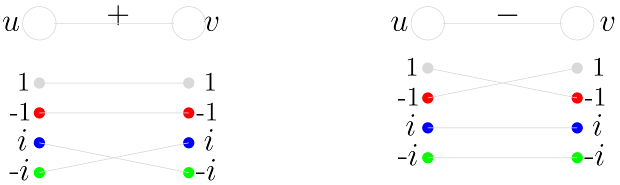


Application in Four color Theorem

Máčajová, Raspaud and Škoviera (2016) conjectures that every signed planar graph MRS-Z-4-colorable. Recently, František Kardoš and Jonathan Narboni disprove the conjecture.

Every signed planar graph is KS-4-colorable?

A complex 4-coloring of a signed graph (G, σ) is a mapping $\varphi : V(G) \to \{1, -1, i, -i\}$ such that $\varphi(u)\varphi(v) \neq \sigma(e)$ for any $uv = e \in E(G)$.



Every signed planar graph is complex 4-colorable?

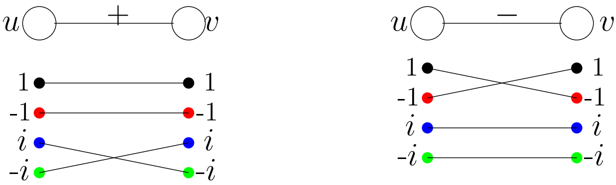
F. Kardoš, J. Narboni, On the 4-color theorem for signed graphs, arXiv:1906.09638v1.
L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint, 2019+

Application in Four color Theorem

Máčajová, Raspaud and Škoviera (2016) conjectures that every signed planar graph MRS-Z-4-colorable. Recently, František Kardoš and Jonathan Narboni disprove the conjecture.

Every signed planar graph is KS-4-colorable?

A complex 4-coloring of a signed graph (G, σ) is a mapping $\varphi : V(G) \to \{1, -1, i, -i\}$ such that $\varphi(u)\varphi(v) \neq \sigma(e)$ for any $uv = e \in E(G)$.



Every signed planar graph is complex 4-colorable?

✠ F. Kardoš, J. Narboni, On the 4-color theorem for signed graphs, arXiv:1906.09638v1.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.

✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint, 2019+

Outline

1. Introduction
   - Coloring
   - Signed coloring
2. Critical permutation set
   - Generalized signed coloring
   - S is Critical
3. Results
   - $S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical
   - $\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_q$ is critical
4. Summary

Generalized signed graph

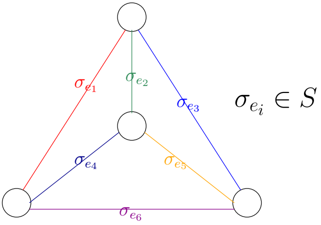
What about graph with more than 2-types of edge?

- S_k is a symmetric group of order k and $S \subseteq S_k$.
- $D(G)$ is a directed graph of G.
- $\sigma : E(D(G)) \to S$ is a mapping which assigns to any $\overrightarrow{e}$ a sign $\sigma_{\overrightarrow{e}}$.

To convenient, we define $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$, and we view G as a symmetric digraph.

Definition (Generalized signed graph)

Assume S is a inverse closed subset of S_k. An *S-signature of G* is a mapping $\sigma : E(G) \to S$ such that $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$. The pair (G, σ) is called an *S-signed graph*.



Generalized signed graph

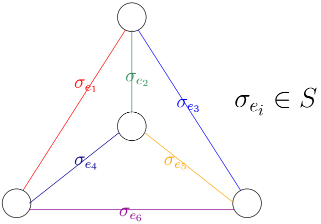
What about graph with more than 2-types of edge?

- S_k is a symmetric group of order k and $S \subseteq S_k$.
- $D(G)$ is a directed graph of G.
- $\sigma : E(D(G)) \to S$ is a mapping which assigns to any $\overrightarrow{e}$ a sign $\sigma_{\overrightarrow{e}}$.

To convenient, we define $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$, and we view G as a symmetric digraph.

Definition (Generalized signed graph)

Assume S is a inverse closed subset of S_k. An *S-signature of G* is a mapping $\sigma : E(G) \to S$ such that $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$. The pair (G, σ) is called an *S-signed graph*.



Generalized signed graph

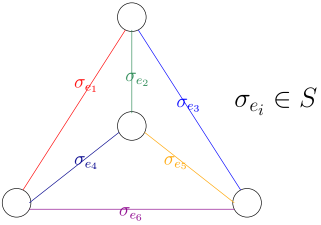
What about graph with more than 2-types of edge?

- S_k is a symmetric group of order k and $S \subseteq S_k$.
- $D(G)$ is a directed graph of G.
- $\sigma : E(D(G)) \to S$ is a mapping which assigns to any $\overrightarrow{e}$ a sign $\sigma_{\overrightarrow{e}}$.

To convenient, we define $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$, and we view G as a symmetric digraph.

Definition (Generalized signed graph)

Assume S is a inverse closed subset of S_k. An *S-signature of G* is a mapping $\sigma : E(G) \to S$ such that $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$. The pair (G, σ) is called an *S-signed graph*.



Generalized signed graph

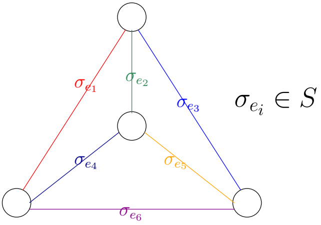
What about graph with more than 2-types of edge?

- S_k is a symmetric group of order k and $S \subseteq S_k$.
- $D(G)$ is a directed graph of G.
- $\sigma : E(D(G)) \to S$ is a mapping which assigns to any $\overrightarrow{e}$ a sign $\sigma_{\overrightarrow{e}}$.

To convenient, we define $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$, and we view G as a symmetric digraph.

Definition (Generalized signed graph)

Assume S is a inverse closed subset of S_k. An S-signature of G is a mapping $\sigma : E(G) \to S$ such that $\sigma_{\overrightarrow{e}} \cdot \sigma_{\overleftarrow{e}} = id$. The pair (G, σ) is called an *S-signed graph*.



Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G, σ), *S-k-coloring of (G, σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G, σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\iff$ k-coloring.
- $S = \{id, (12)(34) \dots (2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\iff$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\iff$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\iff$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\iff$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\iff$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\iff$ group coloring.
- $S = S_k$, S-k-coloring $\iff$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.

✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G, σ), *S-k-coloring of (G, σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G, σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\iff$ k-coloring.
- $S = \{id, (12)(34) \ldots (2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\iff$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\iff$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\iff$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\iff$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\iff$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\iff$ group coloring.
- $S = S_k$, S-k-coloring $\iff$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.
✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G, σ), *S-k-coloring of (G, σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G, σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\Longleftrightarrow$ k-coloring.
- $S = \{id, (12)(34) \dots (2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\Longleftrightarrow$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\Longleftrightarrow$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\Longleftrightarrow$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\Longleftrightarrow$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\Longleftrightarrow$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\Longleftrightarrow$ group coloring.
- $S = S_k$, S-k-coloring $\Longleftrightarrow$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.

✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G, σ), *S-k-coloring of (G, σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G, σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\iff$ k-coloring.
- $S = \{id, (12)(34) \dots (2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\iff$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\iff$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\iff$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\iff$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\iff$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\iff$ group coloring.
- $S = S_k$, S-k-coloring $\iff$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.

✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G,σ), *S-k-coloring of (G,σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G,σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\Longleftrightarrow$ k-coloring.
- $S = \{id, (12)(34)\dots(2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\Longleftrightarrow$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\Longleftrightarrow$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\Longleftrightarrow$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\Longleftrightarrow$ KS-4-coloring;
  - $S = \{(12),(34)\}$, S-4-coloring $\Longleftrightarrow$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\Longleftrightarrow$ group coloring.
- $S = S_k$, S-k-coloring $\Longleftrightarrow$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.
✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G,σ), *S-k-coloring of (G,σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G,σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\Longleftrightarrow$ k-coloring.
- $S = \{id, (12)(34) \dots (2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\Longleftrightarrow$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\Longleftrightarrow$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\Longleftrightarrow$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\Longleftrightarrow$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\Longleftrightarrow$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\Longleftrightarrow$ group coloring.
- $S = S_k$, S-k-coloring $\Longleftrightarrow$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.

✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G, σ), *S-k-coloring of (G, σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G, σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\iff$ k-coloring.
- $S = \{id, (12)(34) \ldots (2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\iff$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\iff$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\iff$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\iff$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\iff$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\iff$ group coloring.
- $S = S_k$, S-k-coloring $\iff$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.
✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

Generalized signed coloring

Definition (Generalized signed coloring)

For an S-signed graph (G,σ), *S-k-coloring of (G,σ)* is a mapping: $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \sigma_e(\varphi(v))$ for any $uv = e \in E(G)$.

G is *S-k-colorable* if for any S-signature σ, (G,σ) has an S-k-coloring.

- $S = \{id\}$, S-k-coloring $\iff$ k-coloring.
- $S = \{id, (12)(34)\ldots(2r\text{-}1\ 2r)\}$,
  $r = \lfloor k/2 \rfloor$, S-k-coloring $\iff$ MRS-Z-k-coloring;
  $r = \lceil k/2 \rceil - 1$, S-k-coloring $\iff$ KS-k-coloring.
  - $S = \{id, (12)(34)\}$, S-4-coloring $\iff$ MRS-Z-4-coloring;
  - $S = \{id, (12)\}$, S-4-coloring $\iff$ KS-4-coloring;
  - $S = \{(12), (34)\}$, S-4-coloring $\iff$ complex-4-coloring.
- $S = \mathbb{Z}_k$, S-k-coloring $\iff$ group coloring.
- $S = S_k$, S-k-coloring $\iff$ DP-k-coloring.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.

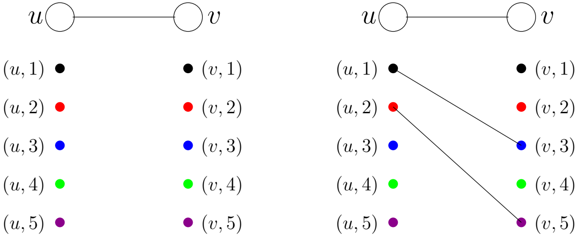
✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

✠ F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs–A non-homongenous analogue of nowwhere-zero, *J. combin. Theory Ser B* 56: 165–182, 1992.

DP-k-coloring

- $S = S_k$, S-k-coloring $\Longleftrightarrow$ DP-k-coloring.

Vertex $v \in V(G)$ is associated with a set of k-colors $\{(v,1),(v,2),\ldots,(v,k)\}$, $uv = e$ is associated with a matching M_e between $\{(u,1),(u,2),\ldots,(u,k)\}$ a-nd $\{(v,1),(v,2),\ldots,(v,k)\}$, restrict colors of u and v for coloring. E.g. $k = 5$:



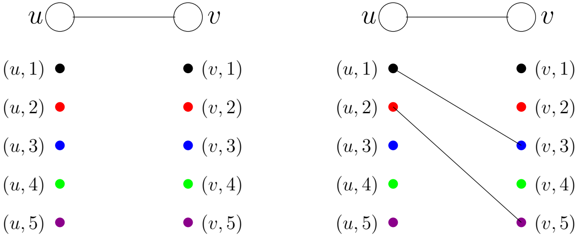
- M_e is consistent and a perfect matching, DP-k-coloring $\Longleftrightarrow$ k-list coloring
- If for any cycle $C = (e_1 e_2 \ldots e_p)$ of (G,σ) satisfies $\sigma_{e_1}\sigma_{e_2}\ldots\sigma_{e_p} = id$, then S-k-coloring $\Longleftrightarrow$ k-list coloring.

✠ Z. Dvořák, L. Postle. Correspondence coloring and its application to list-col oring planar graphs without cycles of lengths 4 to 8. *JCTB*, 129: 38-54, 2018.

DP-k-coloring

- $S = S_k$, S-k-coloring $\Longleftrightarrow$ DP-k-coloring.

Vertex $v \in V(G)$ is associated with a set of k-colors $\{(v,1),(v,2),\ldots,(v,k)\}$, $uv = e$ is associated with a matching M_e between $\{(u,1),(u,2),\ldots,(u,k)\}$ a-nd $\{(v,1),(v,2),\ldots,(v,k)\}$, restrict colors of u and v for coloring. E.g. $k = 5$:



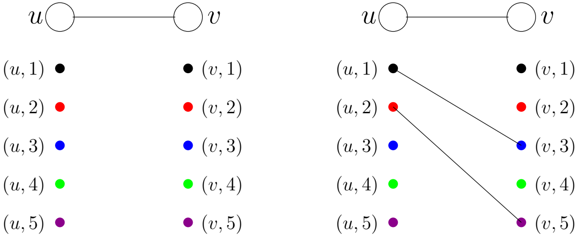
- M_e is consistent and a perfect matching, DP-k-coloring $\Longleftrightarrow$ k-list coloring
- If for any cycle $C = (e_1e_2\ldots e_p)$ of (G,σ) satisfies $\sigma_{e_1}\sigma_{e_2}\ldots\sigma_{e_p} = id$, then S-k-coloring $\Longleftrightarrow$ k-list coloring.

✠ Z. Dvořák, L. Postle. Correspondence coloring and its application to list-col oring planar graphs without cycles of lengths 4 to 8. *JCTB*, 129: 38-54, 2018.

DP-k-coloring

- $S = S_k$, S-k-coloring $\iff$ DP-k-coloring.

Vertex $v \in V(G)$ is associated with a set of k-colors $\{(v,1),(v,2),\ldots,(v,k)\}$, $uv = e$ is associated with a matching M_e between $\{(u,1),(u,2),\ldots,(u,k)\}$ a-nd $\{(v,1),(v,2),\ldots,(v,k)\}$, restrict colors of u and v for coloring. E.g. $k = 5$:



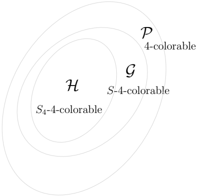
- M_e is consistent and a perfect matching, DP-k-coloring $\iff$ k-list coloring
- If for any cycle $C = (e_1 e_2 \ldots e_p)$ of (G, σ) satisfies $\sigma_{e_1}\sigma_{e_2}\ldots\sigma_{e_p} = id$, then S-k-coloring $\iff$ k-list coloring.

✠ Z. Dvořák, L. Postle. Correspondence coloring and its application to list-col oring planar graphs without cycles of lengths 4 to 8. *JCTB*, 129: 38-54, 2018.

Outline

1. Introduction
   - Coloring
   - Signed coloring
2. Critical permutation set
   - Generalized signed coloring
   - S is Critical
3. Results
   - $S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical
   - $\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_q$ is critical
4. Summary

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable? $S = \{id\}$, for $S \neq \{id\}$?



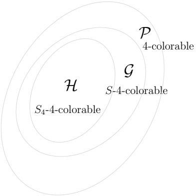
- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable? $S = \{id\}$, for $S \neq \{id\}$?



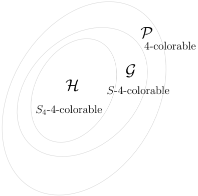
- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

✠ L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
✠ Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable?
  $S = \{id\}$, for $S \neq \{id\}$?



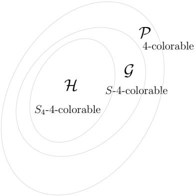
- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable?
  $S = \{id\}$, for $S \neq \{id\}$?



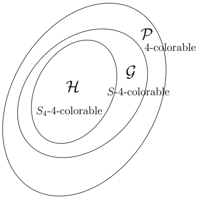
- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable?
  $S = \{id\}$, for $S \neq \{id\}$?



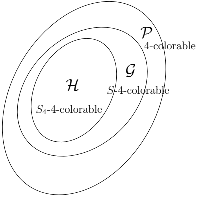
- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
- Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable?
  $S = \{id\}$, for $S \neq \{id\}$?



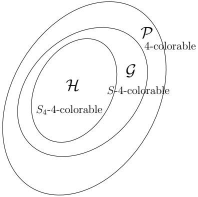
- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
- Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

- Note that not every planar graph is 4-list-colorable.
- Not every planar graph is S_4-4-colorable.
- For which $S \subsetneq S_4$, every planar graph is S-4-colorable?
  $S = \{id\}$, for $S \neq \{id\}$?



- For which $\{id\} \subsetneq S \subsetneq S_4$, $\mathcal{H} \subsetneq \mathcal{G} \subsetneq \mathcal{P}$?
- Find S such that for $S \subsetneq S'$, there exists an S-4-colorable planar graph G which is not S'-4-colorable. $\{id\} \subseteq S \subsetneq S_4$.

Theorem

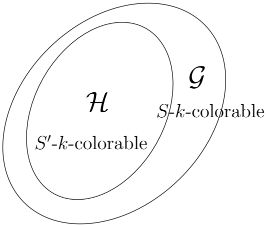
Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

L. Jin, T. Wong, X. Zhu, Colouring of generalized signed planar graphs, arXiv:1811.08584v2.
Y. Jiang, X. Zhu, 4-Colouring of generalized signed planar graphs, preprint.

S is Critical

Now we consider a general k, and general graph.

- S is *trivial*: there exists $i_0 \in [k]$ s. t. $\sigma(i_0) \neq i_0$ for any $\sigma \in S$.
- If $S \subset S_k$ is trivial, then every graph G is S-k-colorable.
- We want non-trivial, inverse closed S and S' s.t. $\{id\} \subseteq S \subsetneq S' \subseteq S_k$.



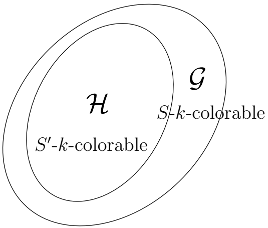
Definition (Critical)

A non-trivial inverse closed subset S of S_k is called *critical* if for any inverse closed subset S' with $S \subsetneq S'$, there is an S-k-colorable graph which is not S'-k-colorable.

S is Critical

Now we consider a general k, and general graph.

- S is *trivial*: there exists $i_0 \in [k]$ s. t. $\sigma(i_0) \neq i_0$ for any $\sigma \in S$.
- If $S \subset S_k$ is trivial, then every graph G is S-k-colorable.
- We want non-trivial, inverse closed S and S' s.t. $\{id\} \subseteq S \subsetneq S' \subseteq S_k$.



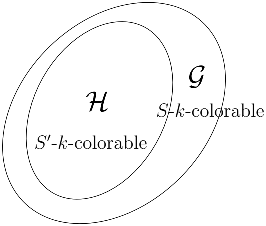
Definition (Critical)

A non-trivial inverse closed subset S of S_k is called *critical* if for any inverse closed subset S' with $S \subsetneq S'$, there is an S-k-colorable graph which is not S'-k-colorable.

S is Critical

Now we consider a general k, and general graph.

- S is *trivial*: there exists $i_0 \in [k]$ s. t. $\sigma(i_0) \neq i_0$ for any $\sigma \in S$.
- If $S \subset S_k$ is trivial, then every graph G is S-k-colorable.
- We want non-trivial, inverse closed S and S' s.t. $\{id\} \subseteq S \subsetneq S' \subseteq S_k$.



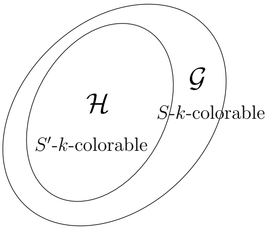
Definition (Critical)

A non-trivial inverse closed subset S of S_k is called *critical* if for any inverse closed subset S' with $S \subsetneq S'$, there is an S-k-colorable graph which is not S'-k-colorable.

S is Critical

Now we consider a general k, and general graph.

- S is *trivial*: there exists $i_0 \in [k]$ s. t. $\sigma(i_0) \neq i_0$ for any $\sigma \in S$.
- If $S \subset S_k$ is trivial, then every graph G is S-k-colorable.
- We want non-trivial, inverse closed S and S' s.t. $\{id\} \subseteq S \subsetneq S' \subseteq S_k$.

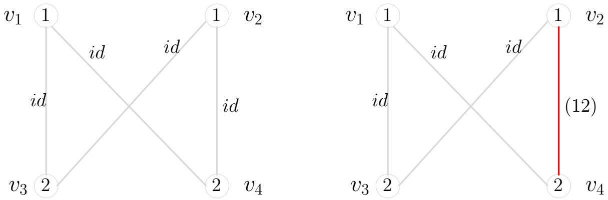


Definition (Critical)

A non-trivial inverse closed subset S of S_k is called *critical* if for any inverse closed subset S' with $S \subsetneq S'$, there is an S-k-colorable graph which is not S'-k-colorable.

S is Critical

- For $k = 2$, $S = \{id\}$ is critical. Even cycle $\notin \mathcal{H}$!



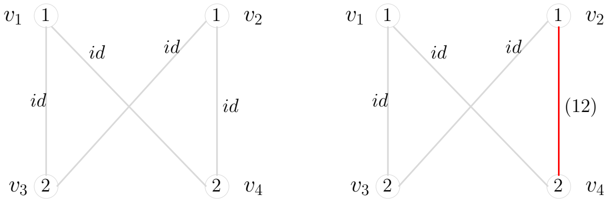
- For $k \geq 3$? Yes!

Lemma

An inverse closed subset S of S_k is critical if and only if there is an S-k-colorable graph G such that for any $\pi \in S_k - S$ with $S' = S \cup \{\pi, \pi^{-1}\}$, G is not S'-k-colorable.

S is Critical

- For $k = 2$, $S = \{id\}$ is critical. Even cycle $\notin \mathcal{H}$!



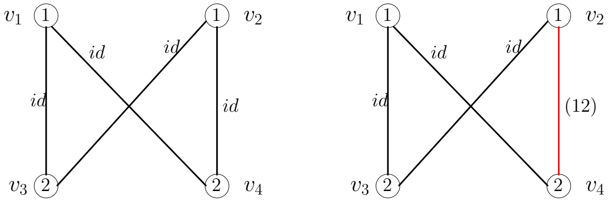
- For $k \geq 3$? Yes!

Lemma

An inverse closed subset S of S_k is critical if and only if there is an S-k-colorable graph G such that for any $\pi \in S_k - S$ with $S' = S \cup \{\pi, \pi^{-1}\}$, G is not S'-k-colorable.

S is Critical

- For $k = 2$, $S = \{id\}$ is critical. Even cycle $\notin \mathcal{H}$!



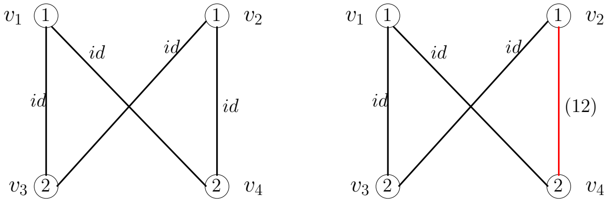
- For $k \geq 3$? Yes!

Lemma

An inverse closed subset S of S_k is critical if and only if there is an S-k-colorable graph G such that for any $\pi \in S_k - S$ with $S' = S \cup \{\pi, \pi^{-1}\}$, G is not S'-k-colorable.

S is Critical

- For $k = 2$, $S = \{id\}$ is critical. Even cycle $\notin \mathcal{H}$!



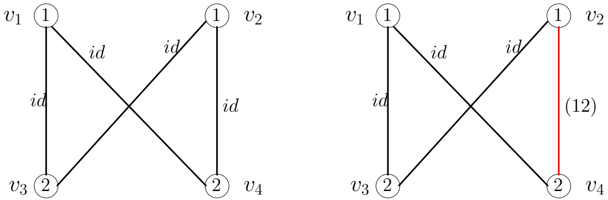
- For $k \geq 3$? Yes!

Lemma

An inverse closed subset S of S_k is critical if and only if there is an S-k-colorable graph G such that for any $\pi \in S_k - S$ with $S' = S \cup \{\pi, \pi^{-1}\}$, G is not S'-k-colorable.

S is Critical

- For $k = 2$, $S = \{id\}$ is critical. Even cycle $\notin \mathcal{H}$!



- For $k \geq 3$? Yes!

Lemma

An inverse closed subset S of S_k is critical if and only if there is an S-k-colorable graph G such that for any $\pi \in S_k - S$ with $S' = S \cup \{\pi, \pi^{-1}\}$, G is not S'-k-colorable.

Outline

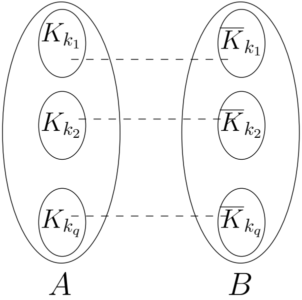
1. Introduction
   - Coloring
   - Signed coloring
2. Critical permutation set
   - Generalized signed coloring
   - S is Critical
3. Results
   - $S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical
   - $\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_q$ is critical
4. Summary

$S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical

Theorem

Assume k, q are two positive integers such that $k_1 + k_2 + \ldots + k_q = k$. Then $S = S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical.

- For $k_1 = k_2 = \ldots = k_q = 1$, then $q = k$ and $S = \{id\}$ is critical.



For $i \in [q]$, $V(K_{k_i}) = \{v_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

For $i \in [q]$, $V(\overline{K}_{k_q}) = \{u_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

$G[A] = K_k$ $\qquad$ $G[B] = K_{k_1,k_2,\ldots,k_q}$

$$E[A,B] = \bigcup_{i=1}^{q} \bigcup_{j=1}^{k_i} \bigcup_{i'=1}^{q} \bigcup_{j'=1}^{k_i} v_{i,j}u_{i',j'} - \bigcup_{i=1}^{q} \bigcup_{j=1}^{k_i} v_{i,j}u_{i,j}$$

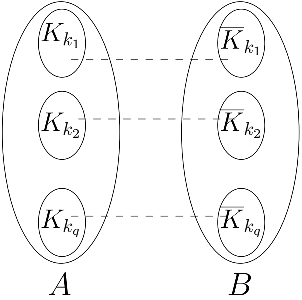
$K_{k_1,k_2,\ldots,k_q}$ complete q-partite graph with partite size are $k_1, k_2, \ldots, k_q$.

$S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical

Theorem

Assume k, q are two positive integers such that $k_1 + k_2 + \ldots + k_q = k$. Then $S = S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical.

- For $k_1 = k_2 = \ldots = k_q = 1$, then $q = k$ and $S = \{id\}$ is critical.



For $i \in [q]$, $V(K_{k_i}) = \{v_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

For $i \in [q]$, $V(\overline{K}_{k_q}) = \{u_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

$G[A] = K_k$ $\qquad$ $G[B] = K_{k_1,k_2,\ldots,k_q}$

$$E[A, B] = \cup_{i=1}^{q} \cup_{j=1}^{k_i} \cup_{i'=1}^{q} \cup_{j'=1}^{k_i} v_{i,j}u_{i',j'} - \cup_{i=1}^{q} \cup_{j=1}^{k_i} v_{i,j}u_{i,j}$$

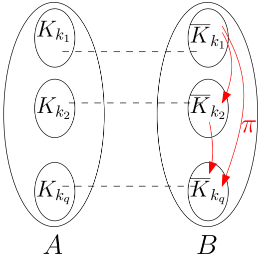
$K_{k_1,k_2,\ldots,k_q}$ complete q-partite graph with partite size are $k_1, k_2, \ldots, k_q$.

$S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical

Theorem

Assume k, q are two positive integers such that $k_1 + k_2 + \ldots + k_q = k$. Then $S = S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical.

- For any signature σ, (G, σ) is S-k-colorable.
- $S' = S \cup \{\pi, \pi^{-1}\}$, there exists a signature σ s.t. (G, σ) is S'-k-colorable.



For $i \in [q]$, $V(K_{k_i}) = \{v_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

For $i \in [q]$, $V(\overline{K}_{k_q}) = \{u_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

$$G[A] = K_k \qquad G[B] = K_{k_1,k_2,\ldots,k_q}$$

$$E[A, B] = \cup_{i=1}^{q} \cup_{j=1}^{k_i} \cup_{i'=1}^{q} \cup_{j'=1}^{k_i} v_{i,j} u_{i',j'} - \cup_{i=1}^{q} \cup_{j=1}^{k_i} v_{i,j} u_{i,j}$$

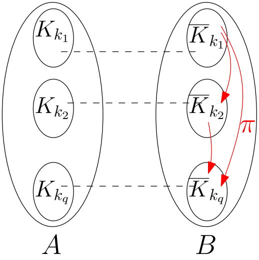
$K_{k_1,k_2,\ldots,k_q}$ complete q-partite graph with partite size are $k_1, k_2, \ldots, k_q$.

$S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical

Theorem

Assume k, q are two positive integers such that $k_1 + k_2 + \ldots + k_q = k$. Then $S = S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical.

- For any signature σ, (G, σ) is S-k-colorable.
- $S' = S \cup \{\pi, \pi^{-1}\}$, there exists a signature σ s.t. (G, σ) is S'-k-colorable.



For $i \in [q]$, $V(K_{k_i}) = \{v_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

For $i \in [q]$, $V(\overline{K}_{k_q}) = \{u_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

$$G[A] = K_k \qquad G[B] = K_{k_1, k_2, \ldots, k_q}$$

$$E[A, B] = \cup_{i=1}^{q} \cup_{j=1}^{k_i} \cup_{i'=1}^{q} \cup_{j'=1}^{k_i} v_{i,j} u_{i',j'} - \cup_{i=1}^{q} \cup_{j=1}^{k_i} v_{i,j} u_{i,j}$$

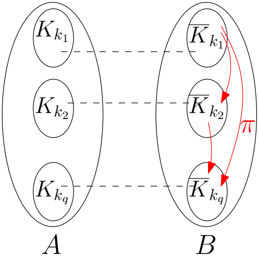
$K_{k_1, k_2, \ldots, k_q}$ complete q-partite graph with partite size are $k_1, k_2, \ldots, k_q$.

$S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical

Theorem

Assume k, q are two positive integers such that $k_1 + k_2 + \ldots + k_q = k$. Then $S = S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical.

- For any signature σ, (G, σ) is S-k-colorable.
- $S' = S \cup \{\pi, \pi^{-1}\}$, there exists a signature σ s.t. (G, σ) is S'-k-colorable.



For $i \in [q]$, $V(K_{k_i}) = \{v_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

For $i \in [q]$, $V(\overline{K}_{k_q}) = \{u_{i,j}\}$, where $j = 1, 2, \ldots, k_i$

$G[A] = K_k$ $\qquad$ $G[B] = K_{k_1,k_2,\ldots,k_q}$

$E[A, B] = \bigcup_{i=1}^{q} \bigcup_{j=1}^{k_i} \bigcup_{i'=1}^{q} \bigcup_{j'=1}^{k_i} v_{i,j}u_{i',j'} - \bigcup_{i=1}^{q} \bigcup_{j=1}^{k_i} v_{i,j}u_{i,j}$

$K_{k_1,k_2,\ldots,k_q}$ complete q-partite graph with partite size are $k_1, k_2, \ldots, k_q$.

Outline

1. Introduction
   - Coloring
   - Signed coloring
2. Critical permutation set
   - Generalized signed coloring
   - S is Critical
3. Results
   - $S_{k_1} \times S_{k_2} \times \ldots \times S_{k_q}$ is critical
   - $\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_q$ is critical
4. Summary

$\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_q$ is critical

Theorem

Assume $[k] = X_1 \cup X_2 \cup \ldots \cup X_q$ *and* $|X_i| = k_i$*. If* $S = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_q$*, where for each* i *either* $\Gamma_i = S_{X_i}$ *or* $|X_i| = 3$ *and* Γ_i *is the subgroup of* S_{X_i} *generated by a cyclic permutation of* X_i*, then* S *is critical.*

Remark: If $|X_i| = 3$, i.e. $X_i = \{a_1, a_2, a_3\}$, then $\Gamma_i = \langle (a_1 a_2 a_3) \rangle$.

For a set X, S_X: the symmtric group of all permutations on X.

Summary

Conjecture

For a positive k, every non-trivial $S \subseteq S_k$ is critical.

- $k = 2$, the conjecture is true.
- $k = 3$?
- $k = 4$?

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- If $S \subseteq \{id, (12), (34), (12)(34)\}$, then every plnanr graph is S-4-colorable?

Summary

Conjecture

For a positive k, every non-trivial $S \subseteq S_k$ is critical.

- $k = 2$, the conjecture is true.
- $k = 3$?
- $k = 4$?

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- If $S \subseteq \{id, (12), (34), (12)(34)\}$, then every plnanr graph is S-4-colorable?

Summary

Conjecture

For a positive k, every non-trivial $S \subseteq S_k$ is critical.

- $k = 2$, the conjecture is true.
- $k = 3$?
- $k = 4$?

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- If $S \subseteq \{id, (12), (34), (12)(34)\}$, then every plnanr graph is S-4-colorable?

Summary

Conjecture

For a positive k, every non-trivial $S \subseteq S_k$ is critical.

- $k = 2$, the conjecture is true.
- $k = 3$?
- $k = 4$?

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- If $S \subseteq \{id, (12), (34), (12)(34)\}$, then every plnanr graph is S-4-colorable?

Summary

Conjecture

For a positive k, every non-trivial $S \subseteq S_k$ is critical.

- $k = 2$, the conjecture is true.
- $k = 3$?
- $k = 4$?

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- If $S \subseteq \{id, (12), (34), (12)(34)\}$, then every plnanr graph is S-4-colorable?

Summary

Conjecture

For a positive k, every non-trivial $S \subseteq S_k$ is critical.

- $k = 2$, the conjecture is true.
- $k = 3$?
- $k = 4$?

Theorem

Up to conjugation, if every planar graph is S-4-colorable, then $S \subseteq \{id, (12), (34), (12)(34)\}$.

- If $S \subseteq \{id, (12), (34), (12)(34)\}$, then every plnanr graph is S-4-colorable?

Thank you for your attention!