Lecture 07. Statistical Inference (I):

Point- and Interval Estimation

統計推論 (I)：點估計與區間估計

- 信賴區間(confidence interval, CI)
- 點估計 point estimation；點估計值 point estimate
- 區間估計 interval estimation
- Sampling distribution of a statistic ⇔ parameter
- Properties of an estimator

  -- Unbiasedness

  -- Consistency
- Two-sided (two-tailed) CI (• One-sided (one-tailed) CI)
- Confidence interval for the population mean with NORMALITY assumption;

  --- 1. variance known

  --- 2. variance unknown: The Student's t-statistic

  --- Graphical illustrations for the meaning of confidence interval
- Confidence interval for the population variance with NORMALITY assumption;
- Confidence interval for the population mean without NORMALITY assumption
- EXAMPLES

常態分布平均值(期望值) μ 之點估計與信賴區間

If $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$,

Q: How to estimate μ ?

A: $\hat{\mu} = \overline{X}$

We call $\overline{X}$ **a point estimator of μ (點估計式)。**

The realization of $\overline{X}$, $\bar{x} = (x_1 + x_2 + \ldots + x_n)/n$, **is the point estimate of μ (點估計值)。**

PROPERTIES

1. Unbiasedness (不偏性)

E($\overline{X}$ **)=μ**

$$E(\overline{X}) = E[(X_1 + X_2 + \ldots + X_n)/n]$$

$$= (1/n)[E(X_1) + E(X_2) + \ldots + E(X_n)]$$

$$= (1/n)[\mu + \mu + \ldots + \mu] = \mu$$

2. Consistency (趨近性；一致性) ★

$\overline{X} \to \mu$**, in probability, or**

Pr(| $\overline{X}$ **-μ|<ε)>1-ε , for arbitrary (usually small) ε>0**

Weak Law of Large Number (弱大數法則)

Sampling distribution of the sample mean

(under 'NORMALITY' assumption) : (I)

- If $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, σ is a **known** value

$$\Longrightarrow \boxed{\bar{X} \sim N(\mu, \sigma^2/n)} \qquad (1)$$

- That is, the sampling distribution of $\bar{X}$ is still Gaussian (normal), for all **n**.

- A suitable standardization [Q1] for (1) gives

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1). \qquad (2)$$

If σ is assumed to be known (but usually it is not), (2) implies:

$$Pr(-1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96) = 0.95,$$

$$Pr(-1.645 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.645) = 0.90, \ldots\ldots$$

Q1: If $Y \sim N(\mu, \sigma^2)$, => $Y - \mu \sim N(0, \sigma^2)$

⇨ $Y - \mu/\sigma \sim N(0,1)$ [why??]

<= $Var[Y - \mu/\sigma] = Var[Y - \mu]/\sigma^2$

$= Var[Y]/\sigma^2 = 1.$

$$\Pr\left(-Z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right) = 1-\alpha, \text{ or } = 100(1-\alpha)\%$$

$\alpha = 0.05$, $Z_{0.025} = 1.96$; $\alpha = 0.10$, $Z_{0.05} = 1.645$;…

We have a two-sided confidence interval

$$\Pr\left(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha \qquad (3)$$

■ Conclusions

μ 之 100(1-α)% CI= $\left(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$ (3')

μ 之 95% CI= $\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}\ ,\ \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right)$

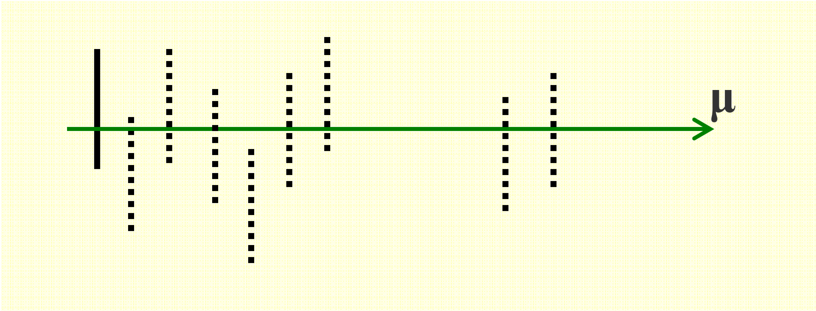
μ 之 90% CI= $\left(\bar{X} - 1.645\frac{\sigma}{\sqrt{n}}\ ,\ \bar{X} + 1.645\frac{\sigma}{\sqrt{n}}\right)$

...

confidence limits

■ 信賴區間的解釋 **(very important !!)**

- **Graphical illustrations for the meaning of confidence interval**



- **If thus constructed 95% CI can have infinite numbers of realizations, then, in average, they will cover the true parameter close to 95 times in 100 replicates.**
  (In a long run !!) (Frequentist's point of view!!)

Q. Why is it necessary to understand the property of confidence interval in the above statement when in practice you will only have a single realization?

■ **Q and A**

□ 在同樣的信心水準(confidence)下，不同的統計方法可能得到不同的信賴區間。不同的信賴區間彼此間是否可以互相比較？

□ 信賴區間的長度是否越短越好？

□ 信賴區間的長度決定於哪些因素？為什麼？

Sampling distribution of the sample mean

(under 'NORMALITY' assumption) : (II)

Variance (σ^2) unknown and the Student's t-statistic

When σ (or σ^2) is unknown, it is not reasonable to report the 95%CI (say) of μ in terms of σ. Instead, the σ appeared in (3) should be replaced by an estimate of σ. Conventionally, we use $s=\hat{\sigma}$, where $s^2=\Sigma(x_i-\bar{x})^2/(n-1)$.

Question:

$$\frac{\bar{\mathbf{x}}-\mu}{\sigma/\sqrt{n}} \sim \mathbf{N}(0,1) \Longrightarrow \frac{\bar{\mathbf{x}}-\mu}{\hat{\sigma}/\sqrt{n}} \sim \mathbf{N}(0,1)???$$

Answer : ★

$$\begin{aligned}\frac{\bar{\mathbf{x}}-\mu}{\hat{\sigma}/\sqrt{n}} &= \frac{\bar{\mathbf{x}}-\mu}{\frac{\sigma}{\sqrt{n}}}\cdot\frac{1}{\frac{\hat{\sigma}}{\sigma}} \\ &= \frac{\bar{\mathbf{x}}-\mu}{\frac{\sigma}{\sqrt{n}}}\cdot\frac{1}{\sqrt{\frac{\sum(\mathbf{x_i}-\bar{\mathbf{x}})^2/\sigma^2}{n-1}}} \\ &= \mathbf{Z}\cdot\frac{1}{\sqrt{\frac{\chi^2_{n-1}}{n-1}}} \equiv \mathbf{t}_{n-1},\end{aligned}$$

$$\text{where } \mathbf{t}^2_\nu \equiv \frac{\chi^2_1/1}{\chi^2_\nu/\nu}$$

□ **The results were derived and published by W. S. Gosset (1908, Biometrika) with the pseudonym of 'Student'.**

□ **Characteristics**

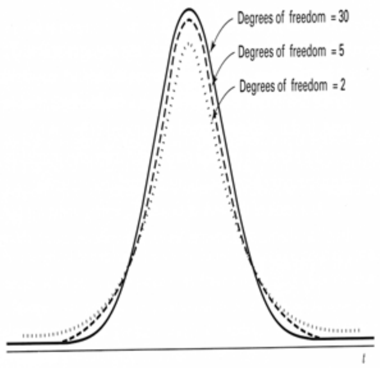


FIGURE 6.3.1 The t distribution for different degrees-of-freedom values.

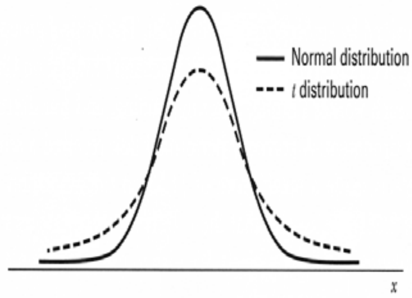


FIGURE 6.3.2 Comparison of normal distribution and t distribution.

□ **Confidence intervals for μ (say, n=21):**

μ 之 100(1-α)% CI=

$$\left(\bar{X} - t_{n-1, \alpha/2}\frac{s}{\sqrt{n}}\ ,\ \bar{X} + t_{n-1, \alpha/2}\frac{s}{\sqrt{n}}\right) \quad (4)$$

95% CI

$$\left(\bar{X} - 2.086\frac{s}{\sqrt{n}}\ ,\ \bar{X} + 2.086\frac{s}{\sqrt{n}}\right)$$

90% CI

$$\left(\bar{X} - 1.725\frac{s}{\sqrt{n}}\ ,\ \bar{X} + 1.725\frac{s}{\sqrt{n}}\right)$$

□ **Example: (See the textbook.)**

Confidence interval for the population mean without NORMALITY assumption: an application of CLT

$X_1, X_2, \ldots, X_n \sim F(\bullet)$, an unknown distribution with $E(X)=\mu$ and $Var(X)=\sigma^2$. Here we derive the approximate $100(1-\alpha)$ % CI of μ:

since $\bar{X} \sim N(\mu, \sigma^2/n)$, or $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

The approximate CI is

$\left(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$; when n gets larger, the approximation gets better. However, since σ is usually unknown, it is replaced by s, the sample standard deviation; in this case, the approximate CI is

$$\left(\bar{X} - Z_{\alpha/2}\frac{s}{\sqrt{n}}, \quad \bar{X} + Z_{\alpha/2}\frac{s}{\sqrt{n}}\right)$$

■ **Example:** Tossing a coin 100 times, and you get 45 'heads'. Let p be the probability of getting a 'head'. Please calculate the 95% CI of p.

Ans: $X_1 \ldots X_{100} \sim Ber(p)$, each $X_i=0$ (tail) or 1 (head). By CLT,

$\dfrac{\bar{x}-p}{\sqrt{Var(\bar{x})}} \sim N(0,1)$. If we use $\widehat{Var}(\bar{x}) = \dfrac{\hat{p}(1-\hat{p})}{n}$, then

$$\Pr\{\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\} = 1-\alpha$$

So, setting alpha=0.05 leads to the 95% CI of p:

(0.45-1.96*sqrt(0.45*0.55/100), 0.45+1.96*sqrt(0.45*0.55/100))

=(0.352, 0.548)