Chapter 7 Finite difference methods for PDE

§7.0 Preliminaries :

① **Theorem 7.1 : (Gerschgoren's Theorem)**

Let A be an $n\times n$ matrix. For each $j=1,\cdots,n$, let

$$C_j=\left\{z\,\middle|\quad |z-A_{jj}|\le\sum_{k\ne j}|A_{jk}|\right\}.$$

Then all the eigenvalues of A are in $\bigcup_{j=1}^{n}C_j$.

Proof : Let $Ax=\lambda x$ where $x=(x_1,\cdots,n)$, and $|x_j|\ge|x_k|$ for all $k=1,\cdots,n$. Then

$$\sum_{k=1}^{n}A_{jk}x_k=\lambda x_j\implies(A_{jj}-\lambda)x_j=\sum_{k\ne j}A_{jk}x_k$$

$$\implies|A_{jj}-\lambda|\,|x_j|\le\sum_{k\ne j}|A_{jk}|\,|x_k|\le|x_j|\sum_{k\ne j}|A_{jk}|$$

$$\implies|A_{jj}-\lambda|\le\sum_{k\ne j}|A_{jk}|\quad\text{since}\quad|x_j|\ne0$$

$$\implies\lambda\in C_j.$$

Problem 87 : Show that if $\mathscr{D}$ is a connected component of $\bigcup_j C_j$ consisting of exactly in m circles, then $\mathscr{D}$ contains exactly m eigenvalues.

② Review on matrix norms and linear algebra

Definition 7.2 : If $x=(x_1,\cdots,x_n)\in\mathbb{C}^n$, define $\|x\|_p=\left(\sum_i^n|x_i|^p\right)^{1/p}$ and $\|x\|_\infty=\max_{1\le i\le n}|x_i|$.

Exercise 88 : $\|x\|_\infty=\lim_{p\to\infty}\|x\|_p$.

Property 7.3 : $\|x\|_p$ is a norm. i.e.

(a) $\|x\|_p\ge0$ and $\|x\|_p=0$ if and only if $x=0$.

(b) $\|\lambda x\|_p=|\lambda|\,\|x\|_p$ for all $\lambda\in\mathbb{R}$ or $\mathbb{C}$.

(c) $\|x+y\|_p=\|x\|_p+\|y\|_p$

Property 7.4 : Any norm is Lipshitz continuous. More precisely, $\big|\,\|x\|-\|y\|\,\big|\le\|x-y\|$.

Proof : $\|x\|=\|y+(x-y)\|\le\|y\|+\|x-y\|\implies\|x\|-\|y\|\le\|x-y\|$.

Similarly, $\|y\|-\|x\|\le\|x-y\|$.

Hence, $\big|\|x\|-\|y\|\big|\le\|x-y\|$

Definition 7.5 : et A be a $n\times n$ complex matrix, define an **operator norm** for A by

$$\|A\|_p=\sup_{x\ne0}\frac{\|Ax\|_p}{\|x\|_p}=\sup_{\|y\|_p=1}\|Ay\|_p$$

<u>Property 7.6</u> : $\|A\|_p$ is a norm (an operator norm), for $1 \le p \le \infty$. i.e.

(a) $\|A\|_p \ge 0$ and $\|A\|_p = 0$ if and only if $A = 0$.

(b) $\|\lambda A\|_p = |\lambda|\ \|A\|_p$, for $\lambda \in \mathbb{C}$.

(c) $\|A+B\|_p \le \|A\|_p + \|B\|_p$ for any 2 $n \times n$ matrice $A,\ B$..

Thus, $\|Ax\|_p \le \|A\|_p\ \|x\|_p$, for $1 \le p \le \infty$.

Exercise 90 : Let $\mathcal{F} = \left\{\ K \mid \|Ay\|_p \le K\|y\|_p \right\}$. Then $\|A\|_p = \inf_{K\in\mathcal{F}} K$.

<u>Property 7.7</u> : $\|AB\|_p \le \|A\|_p\ \|B\|_p$ for all $n \times n$ matrices A, B .

<u>Proof :</u> $\|ABy\|_p = \|A(By)\|_p \le \|A\|_p\ \|By\|_p \le \|A\|_p\|B\|_p\|y\|_p$.

$$\therefore \quad \sup_{\|y\|_p=1} \frac{\|ABy\|_p}{\|y\|_p} \le \|A\|_p\ \|B\|_p.$$

<u>Property 7.8</u> :

(a) If $D = \begin{bmatrix} d_1 & 0 & \cdots & \cdots \\ 0 & \ddots & & \cdots\ 0 \\ \vdots & & & \ddots\ 0 \\ 0 & & & \cdot\cdot\ d_n \end{bmatrix} = \mathrm{diag}(d_1\,, \cdots, d_n)$ then $\|D\|_p = \max_i |d_i|$.

(b) $\|A\|_1 = \max_k \sum_j |A_{jk}|$.

(c) $\|A\|_\infty = \max_j \sum_k |A_{jk}|$

How about $\|A\|_2$?

<u>Definition 7.9</u> : Let A be a $n \times n$ matrix. Set

$\sigma(A) =$ the set of eigenvalues of A

and

$$\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|.$$

<u>Property 7.10</u> : $\|A\|_p \ge \rho(A)$ for all $p \in [1,\infty]$.

<u>Proof</u> : If $Ax = \lambda\, x$ and $\|x\|_p = 1$ then $|\lambda| = \|Ax\|_p \le \|A\|_p$.

$\therefore \quad \rho(A) \le \|A\|_p$.

<u>Property 7.11</u> : If A is normal (i.e. $AA^* = A^*A$) then $\|A\|_2 = \rho(A)$.

In any case, $\|A\|_2 = \rho(A^*A)^{1/2} \le \|A^*A\|_2^{1/2} \le \|A^*\|_2\ \|A\|_2$.

Exercise 91 : Find sharp constants C_1 and C_2, depending on n such that

$$C_1\|x\|_\infty \le \|x\|_2 \le C_2\|x\|_\infty$$

<u>Definition</u> 7.12 : The *o* , *O* notations: $c \in [-\infty,\infty]$

(a) $f(x) = o(g(x))$ as $x \to c$ if $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = 0$,.

(b) $f(x) = O(g(x))$ as $x \to c$ if $\exists$ a constant K such that $|f(x)| \le K\, g(x)$, for all x near c.

(note that here c can be replaced by $c^{\pm}$)

Theorem 7.12.A : Taylor's formula with Remainder

Let f be a function whose $(n+1)$st derivative $f^{(n+1)}$(x) exists for each x in an open interval I containing a. Then, for each x in I,

$$f(x) = f(a) + f'(a)(x-1) + \frac{f''}{2!}(x-a)^2 \cdots + \frac{f^{(n)}}{n!}(x-a)^n + R_n(x)$$

where the remainder (or error) $R_n(x)$ is given by the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

and c is some point between x and a.

Theorem 7.12.B : Taylor's Theorem

Let f be a function with derivatives of all orders in some interval $(a-r, a+r)$. The Taylor series

$$f(x) = f(a) + f'(a)(x-1) + \frac{f''(a)}{2!}(x-a)^2 \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

represents the function f on the interval $(a-r, a+r)$ if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

where the remainder (or error) $R_n(x)$ is given by the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

and c is some point in the interval $(a-r, a+r)$.

Theorem 7.12.C : Property on operator normed

Let $(X, \|\cdot\|)$ be a complete normed space (Banach space: every Cauchy sequence is convergent.) and $T : X \to X$ an operator with the operator norm $\|T\|_X < 1$. Define $T^0 = I$, $T^2(x) = T(T(x))$ and $T^n(x) = T(T^{n-1}(x))$. Then

(a) $S(x) = \lim_{n\to\infty} \sum_{k=0}^{n} T^k(x)$ is well-defined with $\|S\|_X = \left(1 - \|T\|_X\right)^{-1}$.

(b) $I - T$ is invertible.

(c) $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k(n)$.

(d) $\left\|(I-T)^{-1}\right\|_X = \left(1 - \|T\|_X\right)^{-1}$.