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Supplementary Material for "Finite Mixtures of Multivariate Scale-Shape Mixtures of Skew-normal Distributions"

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This supporting information contains additional results for the simulation studies.

S1. Additional Figures for Simulation 1

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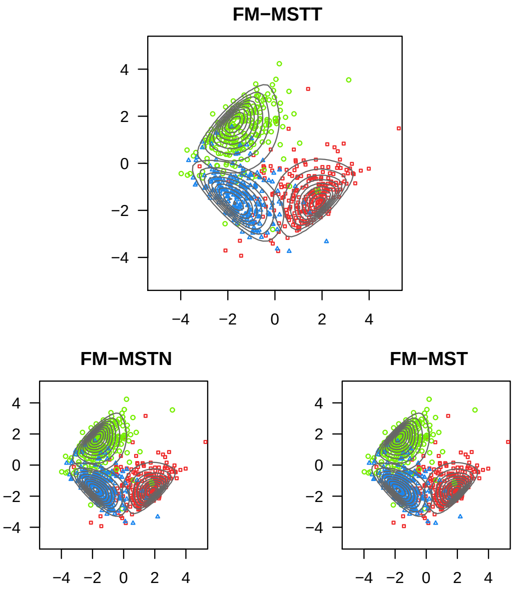

Fig. S.1 A simulated sample ($n = 1500$) from the FM-MSTN model and the respective density contours.

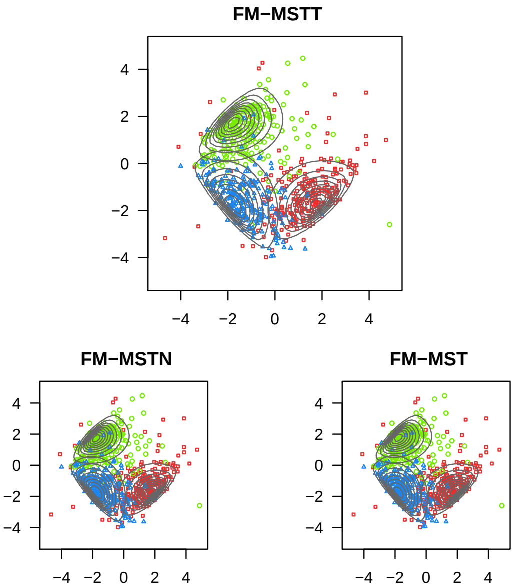

Fig. S.2 A simulated sample ($n = 1500$) from the FM-MST model and the respective density contours.

S2. Additional Results for Simulation 2

A similar experiment as in Section 4.2 is undertaken for two-component FM-MSGLN and FM-MSSN models to examine the finite samples properties of the ML estimators based on the proposed ECM algorithm. For drawing MC samples from the two models, the simulation settings are as follows:

(i) For the FM-MSGLN model, the presumed parameters are

$$\pi = 0.5,\ \boldsymbol{\xi}_1 = (-3,0)^\top,\ \boldsymbol{\xi}_2 = (-5,2)^\top,\ \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{I}_2,$$
$$\boldsymbol{\lambda}_1 = (3,-4)^\top,\ \boldsymbol{\lambda}_2 = (2,3)^\top,\ \alpha_1 = 8,\ \alpha_2 = 6.$$

(ii) For the FM-MSSN model, the presumed parameters are

$$\pi = 0.5,\ \boldsymbol{\xi}_1 = (3,1)^\top,\ \boldsymbol{\xi}_2 = (-1,2)^\top,\ \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{I}_2,$$
$$\boldsymbol{\lambda}_1 = (-3,2)^\top,\ \boldsymbol{\lambda}_2 = (3,-2)^\top,\ q_1 = 5,\ q_2 = 3.$$

Figure S.3 displays the scatter-contour plots of one simulation case ($n = 1000$) for samples randomly drawn from the FM-MSGLN and FM-MSSN models.

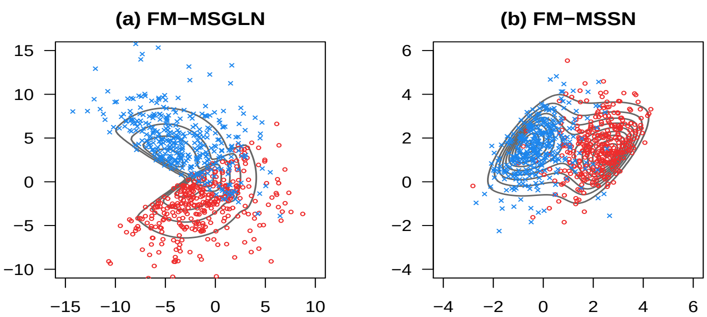

Fig. S.3 Scatter-contour plots of one simulation case ($n = 1000$) for samples randomly drawn from the FM-MSGLN and FM-MSSN models.

Analogous to the FM-MSTT case already presented in Section 4.2, numerical results summarized in Tables S.1 and S.2 indicate that the proposed algorithm can provide reliable and accurate parameter estimates under the scenario of FM-MSGLN and FM-MSSN models when n is sufficiently large.

Table S.1 Simulation results of the FM-MSGLN model for assessing the consistency of parameter estimates and standard errors with various sample sizes. The notation f_{ir} denotes the elements contained in vech($\boldsymbol{\Sigma}_i^{1/2}$) for $i = 1,2$ and $r = 1,2,3$.

Sample Size (n)	Component (i)	Measure	Parameter								
			π	ξ_{i1}	ξ_{i2}	f_{i1}	f_{i2}	f_{i3}	λ_{i1}	λ_{i2}	α_i
500	1	AB	0.022	0.335	0.309	0.482	0.067	0.469	0.981	1.331	6.099
		MSE	50.001	0.178	0.335	0.387	0.010	0.363	1.379	2.578	82.931
		Emp Std	0.030	0.422	0.571	0.605	0.098	0.589	0.963	1.289	8.533
		IM Std	0.033	0.424	0.342	0.378	0.041	0.372	1.969	2.594	19.910
	2	AB	–	0.349	0.288	0.473	0.087	0.430	0.726	1.032	5.235
		MSE	–	0.196	0.341	0.440	0. 038	0.309	0.829	1.753	71.460
		Emp Std	–	0.443	0.584	0.649	0.192	0.550	0.911	1.325	7.702
		IM Std	–	0.419	0.294	0.358	0.049	0.329	1.415	2.090	18.617
1000	1	AB	0.014	0.232	0.182	0.421	0.050	0.413	0.807	1.084	3.842
		MSE	3e–4	0.087	0.054	0.287	0.004	0.263	0.918	1.806	30.817
		Emp Std	0.018	0.295	0.227	0.528	0.066	0.508	0.715	0.920	5.360
		IM Std	0.023	0.295	0.234	0.309	0.031	0.304	1.220	1.615	8.033
	2	AB	–	0.227	0.165	0.404	0.503	0.369	0.522	0.741	2.859
		MSE	–	0.086	0.045	0.279	0.005	0.211	0.418	0.833	18.654
		Emp Std	–	0.293	0.212	0.528	0.072	0.459	0.646	0.913	4.076
		IM Std	–	0.291	0.205	0.288	0.037	0.271	0.813	1.194	5.626
2000	1	AB	0.011	0.163	0.129	0.344	0.035	0.320	0.685	0.944	2.693
		MSE	2e–4	0.043	0.027	0.179	0.002	0.148	0.649	1.217	12.990
		Emp Std	0.012	0.207	0.158	0.413	0.047	0.382	0.506	0.671	3.565
		IM Std	0.016	0.207	0.164	0.211	0.021	0.206	0.805	1.068	3.321
	2	AB	–	0.174	0.121	0.258	0.040	0.239	0.339	0.487	1.767
		MSE	–	0.046	0.023	0.113	0.003	0.088	0.178	0.364	7.175
		Emp Std	–	0.214	0.151	0.336	0.050	0.292	0.421	0.602	3.617
		IM Std	–	0.205	0.145	0.198	0.025	0.185	0.514	0.745	3.516
4000	1	AB	0.006	0.134	0.093	0.209	0.027	0.205	0.674	0.905	1.595
		MSE	1e–4	0.026	0.013	0.068	0.001	0.069	0.553	0.995	4.185
		Emp Std	0.007	0.160	0.115	0.254	0.033	0.258	0.347	0.439	2.056
		IM Std	0.011	0.145	0.115	0.140	0.015	0.139	0.525	0.704	2.275
	2	AB	–	0.115	0.077	0.172	0.031	0.166	0.218	0.287	1.004
		MSE	–	0.019	0.009	0.047	0.001	0.044	0.071	0.133	1.690
		Emp Std	–	0.139	0.093	0.218	0.038	0.210	0.263	0.360	1.289
		IM Std	–	0.145	0.102	0.134	0.018	0.124	0.342	0.496	1.606

Table S.2 Simulation results of FM-MSSN model for assessing the consistency of parameter estimates and standard errors with various sample sizes. The notation f_{ir} denotes the elements contained in vech$(\boldsymbol{\Sigma}_i^{1/2})$ for $i = 1,2$ and $r = 1,2,3$.

Sample Size (n)	Component (i)	Measure	Parameter								
			π	ξ_{i1}	ξ_{i2}	f_{i1}	f_{i2}	f_{i3}	λ_{i1}	λ_{i2}	q_i
500	1	AB	0.054	0.190	0.347	0.468	0.313	0.322	1.629	1.403	8.829
		MSE	0.004	0.073	0.229	0.414	0.194	0.182	4.178	3.173	294.251
		Emp Std	0.065	0.263	0.436	0.639	0.364	0.420	2.039	1.748	15.554
		IM Std	0.047	0.102	0.120	0.100	0.048	0.074	1.070	0.775	51.703
	2	AB	–	0.153	0.225	0.717	0.272	0.246	1.409	0.899	5.492
		MSE	–	0.041	0.103	1.159	0.175	0.110	3.317	1.446	211.077
		Emp Std	–	0.203	0.306	0.951	0.373	0.314	1.715	1.202	13.570
		IM Std	–	0.090	0.102	0.110	0.047	0.060	1.184	0.736	27.890
1000	1	AB	0.034	0.108	0.183	0.313	0.179	0.214	1.012	0.772	4.312
		MSE	0.002	0.025	0.083	0.175	0.084	0.104	1.808	1.128	101.436
		Emp Std	0.043	0.156	0.273	0.420	0.262	0.324	1.346	1.052	9.559
		IM Std	0.030	0.066	0.077	0.073	0.031	0.048	0.694	0.496	14.012
	2	AB	–	0.093	0.134	0.390	0.127	0.162	0.784	0.544	1.386
		MSE	–	0.014	0.034	0.418	0.040	0.049	1.078	0.543	27.081
		Emp Std	–	0.119	0.176	0.602	0.189	0.217	1.012	0.733	5.084
		IM Std	–	0.059	0.068	0.073	0.030	0.041	0.693	0.455	2.882
2000	1	AB	0.023	0.066	0.096	0.185	0.069	0.101	0.592	0.431	1.256
		MSE	0.001	0.008	0.021	0.058	0.008	0.017	0.642	0.354	5.457
		Emp Std	0.029	0.090	0.144	0.242	0.090	0.128	0.796	0.595	2.230
		IM Std	0.019	0.043	0.051	0.052	0.021	0.033	0.481	0.347	1.581
	2	AB	–	0.054	0.075	0.173	0.061	0.086	0.479	0.337	0.522
		MSE	–	0.005	0.010	0.075	0.009	0.013	0.409	0.191	7.572
		Emp Std	–	0.070	0.099	0.272	0.093	0.113	0.628	0.437	2.736
		IM Std	–	0.040	0.046	0.047	0.019	0.027	0.456	0.318	0.781
4000	1	AB	0.017	0.046	0.055	0.131	0.047	0.082	0.388	0.250	0.067
		MSE	4e–4	0.003	0.005	0.027	0.004	0.011	0.231	0.099	1.175
		Emp Std	0.021	0.058	0.073	0.163	0.061	0.106	0.478	0.314	1.036
		IM Std	0.013	0.030	0.035	0.037	0.015	0.023	0.332	0.242	0.755
	2	AB	–	0.033	0.045	0.112	0.043	0.067	0.260	0.202	0.161
		MSE	–	0.002	0.004	0.020	0.003	0.006	0.111	0.066	0.042
		Emp Std	–	0.043	0.061	0.140	0.053	0.080	0.332	0.257	0.203
		IM Std	–	0.028	0.032	0.033	0.013	0.019	0.311	0.220	0.218