Chapter 12

On the Limiting Properties of Binomial and Multinomial Option Pricing Models: Review and Integration

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In this chapter we reviewed two well known papers on binomial option pricing. They are the papers by Cox, Ross, and Rubinstein (1979) and Rendleman and Barter (1979). We have also reviewed the multinomial extension as given in Madan, Milne, and Shefrin (1989). In this review we will give detailed derivations of the limiting results, some of which are new and could not be found from the original papers. It will enable the statistician interested in option pricing models to understand the basic ingredients of the option pricing models. It will also provide some insights of the mathematical details for the finance professional. Thus, it is our belief that the chapter is useful for serious readers in binomial and multinomial option pricings.

Keywords: Black–Scholes formula; central limit theorem; multivariate normality; risk neutrality; Taylor series expansion.

1. Introduction

The main purpose of this chapter is to review two important binomial option pricing model papers and one multinomial option pricing model with an emphasis of the limiting properties of the results when the lapsed time between successive stock price changes tends to zero. This will enable the statistician interested in option pricing models to understand some basic ingredients of the option pricing models. It will also facilitate the understanding of the statistical aspects of the option pricing models for the finance professional. Some of the detailed derivations are new and could not be found from the original papers. Hence, the chapter is useful for serious readers interested in the binomial option pricing model and the multinomial option pricing model and their relationship to the celebrated Black–Scholes model.

2. The Binomial Option Pricing Model of Cox, Ross, and Rubinstein

In this section we will concentrate on the limiting behavior of the binomial option pricing model proposed by Cox, Ross, and Rubinstein (CRR, 1979).

2.1. *The binomial option pricing formula of CRR*

Let S be the current stock price, K be the strike price and $R - 1$ be the riskless rate. It is assumed that the stock follows a binomial process, from one period to the next it can only go up by a factor of u with probability p or go down by a factor of d with probability $1 - p$. After n periods to maturity, CRR showed that the option price C is:

$$C = \frac{1}{R^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \mathrm{Max}[0, u^k d^{n-k} S - K] \quad (1)$$

An alternative expression for C, which is easier to evaluate, is

$$\begin{aligned} C &= S\left[\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{u^k d^{n-k}}{R^n}\right] \\ &\quad - \frac{K}{R^n}\left[\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}\right] \\ &= S\mathrm{B}(m; n, p') - \frac{K}{R^n}\mathrm{B}(m; n, p) \end{aligned} \quad (2)$$

Where $p' = p\frac{u}{R}$, $\mathrm{B}(m; n, p) = \sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k}$ and m is the minimum number of upward stock movements necessary for the option to terminate *in the money*, i.e., m is the minimum value of k in (1) such that $u^m d^{n-m} S - X > 0$.

2.2. *Limiting case*

We now show that the binomial option pricing formula as given in (2) will converge to the celebrated Black–Scholes option pricing model. The Black–Scholes formula is

$$C = SN(d_1) - e^{-rt}KN(d_1 - \sigma\sqrt{t}) \tag{3}$$

Where r be the continuous compound interest such that $R^n = e^{rt}$ as $n \to \infty$, and

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}} \tag{4}$$

In order to show the limiting result that the binomial option pricing formula converges to the continuous version of Black–Scholes option pricing formula, we suppose that h represents the elapsed time between successive stock price changes. Thus, if t is the fixed length of calendar time to expiration, and n is the total number of periods each with length h, then $h = t/n$. As the trading frequency increases, h will get closer to zero. When $h \to 0$, this is equivalent to $n \to \infty$.

Let S^* be the stock price at the end of the nth period with the initial price S. If there are j upward moves during the n periods, then

$$\log\left(\frac{S^*}{S}\right) = j\log(u) + (n-j)\log(d) = j\log\left(\frac{u}{d}\right) + n\log(d) \tag{5}$$

Since j is the realization of a binomial random variable with probability of a success being p, we have expectation of $\log(S^*/S)$

$$E\left(\log\left(\frac{S^*}{S}\right)\right) = n\left[p\log(u/d) + \log(d)\right] = \hat{\mu}n \tag{6}$$

And its variance

$$\mathrm{Var}\left(\log\left(\frac{S^*}{S}\right)\right) = n\left[\log(u/d)\right]^2 p(1-p) = \hat{\sigma}^2 n \tag{7}$$

Since we divide up our original longer time period t into many shorter subperiods of length h so that $t = nh$, our procedure calls for making n longer, while keeping the length t fixed. In the limiting process we would want the mean and the variance of the continuously compounded log rate of return of the assumed stock price movement to coincide with that of actual stock price as $n \to \infty$. Let the actual values of $\hat{\mu}n$ and $\hat{\sigma}^2 n$ be μt and $\sigma^2 t$, respectively. It can be shown that if we set

$$u = e^{\sigma\sqrt{t/n}}, \quad d = e^{-\sigma\sqrt{t/n}}, \quad p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{t}{n}} \tag{8}$$

Then

$$\hat{\mu}n \to \mu t \quad \text{and} \quad \hat{\sigma}^2 n \to \sigma^2 t \quad \text{as } n \to \infty$$

Lemma 1 (Lyapounov's Condition). *Suppose $X_1, X_2, \cdots$ are independent and uniformly bounded with* $E(X_i) = 0$, $Y_n = X_1 + \ldots + X_n$ and $s_n^2 = E(Y_n^2) = \mathrm{Var}(Y_n)$. If

$$\lim \sum_{k=1}^{n} \frac{1}{s_n^{2+\delta}} E|X_k|^{2+\delta} = 0, \quad \textit{for some } \delta > 0$$

Then

$$\frac{Y_n}{s_n} \xrightarrow{d} N(0,1), \quad \textit{as } n \to \infty$$

Theorem 1. *If*

$$\frac{p|\log(u) - \hat{\mu}|^3 + (1-p)|\log(d) - \hat{\mu}|^3}{\hat{\sigma}^3\sqrt{n}} \to 0 \textit{ as } \quad n \to \infty \tag{9}$$

Then

$$\Pr\left(\frac{\log(S^*/S) - \hat{\mu}n}{\hat{\sigma}\sqrt{n}} \le z\right) \to N(z) \tag{10}$$

where N(z) denotes the cumulative standard normal distribution function.

Proof. Since

$$\begin{aligned} p|\log(u) - \hat{\mu}|^3 &= p|\log(u) - p\log(u/d) - \log(d)|^3 \\ &= p(1-p)^3|\log(u/d)|^3 \end{aligned}$$

And

$$\begin{aligned} (1-p)|\log(d) - \hat{\mu}|^3 &= (1-p)|\log(d) - p\log(u/d) - \log(d)|^3 \\ &= p^3(1-p)|\log(u/d)|^3 \end{aligned}$$

We have

$$p|\log(u) - \hat{\mu}|^3 + (1-p)|\log(d) - \hat{\mu}|^3 = p(1-p)[(1-p)^2 + p^2]|\log(u/d)|^3$$

Thus,

$$\frac{p|\log(u)-\hat{\mu}|^3+(1-p)|\log(d)-\hat{\mu}|^3}{\hat{\sigma}^3\sqrt{n}}$$
$$=\frac{p(1-p)[(1-p)^2+p^2]|\log(u/d)|^3}{[\sqrt{p(1-p)}\log(u/d)]^3\sqrt{n}}$$
$$=\frac{(1-p)^2+p^2}{\sqrt{np(1-p)}}\to 0 \quad \text{as } n\to\infty$$

It is noted that the condition (9) is a special case of the Lyapounov's condition in Lemma 1 when $\delta = 1$. Hence, (10) is established by Lemma 1.

We will next show that the binomial option pricing model as given in (2) will indeed coincide with the Black–Scholes option pricing formula as given in (3). In order to show the limiting result, we need to show that as $n\to\infty$,

$$\mathrm{B}(m;n,p')\to N(d_1) \quad \text{and} \quad \mathrm{B}(m;n,p)\to N(d_1-\sigma\sqrt{t})$$

We will only show the second convergence results, as the same argument will hold true for the first convergence. From the definition of $\mathrm{B}(m;n,p)$, it is clear that

$$1-\mathrm{B}(m;n,p)=\Pr(j\le m-1)$$
$$=\Pr\left(\frac{j-np}{\sqrt{np(1-p)}}\le\frac{m-1-np}{\sqrt{np(1-p)}}\right) \tag{11}$$

From (5) and the definitions for $\hat{\mu}$ and $\hat{\sigma}^2$, we have

$$\frac{j-np}{\sqrt{np(1-p)}}=\frac{\log(S^*/S)-\hat{\mu}n}{\hat{\sigma}\sqrt{n}} \tag{12}$$

Also, from the binomial option pricing formula we have

$$m-1=\frac{\log\left(\frac{K}{Sd^n}\right)}{\log(u/d)}-\varepsilon$$

Where ε is a real number between 0 and 1.

It is easy to show that

$$\frac{m-1-np}{\sqrt{np(1-p)}} = \frac{\log(K/S) - \hat{\mu}n - \varepsilon\log(u/d)}{\hat{\sigma}\sqrt{n}}$$

In order to apply the central limit theorem, we have to evaluate the asymptotic results of $\hat{\mu}n$, $\hat{\sigma}^2n$ and $\log(u/d)$. It is clear that

$$\hat{\mu}n \to (r - \sigma^2/2)t, \quad \hat{\sigma}^2 n \to \sigma^2 t, \quad \text{and} \quad \log(u/d) \to 0$$

Hence

$$\frac{\log(K/S) - \hat{\mu}n - \varepsilon\log(u/d)}{\hat{\sigma}\sqrt{n}} \to z = \frac{\log(K/S) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}$$

Using the fact that $1 - N(z) = N(-z)$, we have

$$\mathrm{B}(m; n, p) \to N(-z) = N(d_1 - \sigma\sqrt{t})$$

Where d_1 is given in (4).

Similar argument holds for $\mathrm{B}(m; n, p')$, and hence we completed the proof that the binomial option pricing formula as given in (2) includes the Block–Scholes option pricing formula as a limiting case.

3. The Binomial Option Pricing of Rendleman and Bartter

In this section we will present an alternative binomial option model proposed by Rendleman and Barter (RB, 1979), which was independently developed and published in the same year as CRR (1979).

3.1. *The binomial option pricing formula of RB*

Similar to option pricing formula in Eq. (2) for the binomial option pricing model of CRR, we have the following binomial option C with

n periods,

$$
\begin{aligned}
C &= S\sum_{i=m}^{n}\binom{n}{i}\left[\frac{(R-H^-)H^+}{(H^+-H^-)R}\right]^i\left[\frac{(H^+-R)H^-}{(H^+-H^-)R}\right]^{n-i} \\
&\quad -\frac{K}{R^n}\sum_{i=m}^{n}\binom{n}{i}\left[\frac{R-H^-}{H^+-H^-}\right]^i\left[\frac{H^+-R}{H^+-H^-}\right]^{n-i} \\
&= S\mathrm{B}(m;n,p')-\frac{K}{R^n}\mathrm{B}(m;n,p)
\end{aligned}
\tag{13}
$$

Where H^+ and H^- represent the returns per dollar invested in the stock if the price *rises*, called the "+" state, and *falls*, called the "−" state, respectively, and

$$
\begin{aligned}
m &= 1+\mathrm{INT}\left[\frac{\log(K/S)-n\cdot\log(H^-)}{\log(H^+)-\log(H^-)}\right] \\
p &= \frac{R-H^-}{H^+-H^-} \\
p' &= \frac{(R-H^-)H^+}{(H^+-H^-)R}=p\frac{H^+}{R}
\end{aligned}
\tag{14}
$$

Here $R-1$ in (13) denotes the riskless interest rate over one period, as in CRR.

3.2. *The limiting case*

Before establishing the convergence of the binomial option pricing formula to the Black–Scholes option pricing formula, we first consider the first two moments of $Y=\log(S^*/S)=\sum_{i=1}^{n}X_i$, where

$$
X_i=\begin{cases} h^+=\log(H^+) & \text{with probability} \quad \theta \\ h^-=\log(H^-) & \text{with probability} \quad 1-\theta \end{cases}
$$

It is noted that random variable Y is the sum of the log-returns over the n periods in which the probability of a price rise is θ and a price fall is $1-\theta$. Also, the movements of the price changes are independent.

Then,

$$\mu_y = \mathrm{E}(\log(S^*/S)) = \mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = n[(h^+ - h^-)\theta + h^-]$$

$$\sigma_y^2 = \mathrm{Var}(\log(S^*/S)) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = n(h^+ - h^-)^2\theta(1-\theta)$$

For the formulae for μ_y and σ_y^2, we have

$$\frac{\mu_y}{n} = (h^+ - h^-)\theta + h^-$$

$$\frac{\sigma_y^2}{n} = (h^+ - h^-)^2\theta(1-\theta)$$

Implying that

$$h^+ = \frac{\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{1-\theta}{\theta}}$$

$$h^- = \frac{\mu_y}{n} - \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}}$$

Since $h^+ = \log(H^+)$ and $h^- = \log(H^-)$, we obtain

$$H^+ = \exp\left\{\frac{\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{1-\theta}{\theta}}\right\}$$

$$H^- = \exp\left\{\frac{\mu_y}{n} - \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}}\right\}$$

The formulae for H^+ and H^- are useful in deriving the asymptotic binomial option pricing formula as $n \to \infty$. From (2), the call option price can be stated as

$$C = S\mathrm{B}(m; n, p') - \frac{K}{R^n}\mathrm{B}(m; n, p).$$

It is well known that the binomial distribution converges to the normal distribution as $n \to \infty$. Thus, the binomial option price C becomes

$$C = S[N(Z_1') - N(Z_1)] - Ke^{-rt}[N(Z_2') - N(Z_2)], \quad \text{as } n \to \infty$$

Where

$$Z_1 = \frac{m - np'}{\sqrt{np'(1-p')}}, \quad Z_1' = \frac{n - np'}{\sqrt{np'(1-p')}}$$

$$Z_2 = \frac{m - np}{\sqrt{np(1-p)}}, \quad Z_2' = \frac{n - np}{\sqrt{np(1-p)}}$$

It is trivial to see that as $n \to \infty$, Z_1' and Z_2' will both tend to ∞. We will next consider $\lim_{n\to\infty} Z_1$ and $\lim_{n\to\infty} Z_2$. Substituting $H^+ = e^{\mu_y/n + (\sigma_y/\sqrt{n})\sqrt{(1-\theta)/\theta}}$, $H^- = e^{\mu_y/n - (\sigma_y/\sqrt{n})\sqrt{\theta/(1-\theta)}}$ and m into Z_1 and Z_2 will lead to

$$\begin{aligned} Z_1 &= \frac{1 + \text{INT}\left[\frac{\log(K/S) - \mu_y + \sigma_y\sqrt{n}\sqrt{\theta/(1-\theta)}}{\sigma_y\sqrt{n\theta(1-\theta)}}\right] - np'}{\sqrt{np'(1-p')}} \\ &\approx \frac{\log(K/S) - \mu_y}{\sigma_y\sqrt{p'(1-p')/\theta(1-\theta)}} + \frac{\sqrt{n}(\theta - p')}{\sqrt{p'(1-p')}} \end{aligned} \tag{15}$$

And

$$\begin{aligned} Z_2 &= \frac{1 + \text{INT}\left[\frac{\log(K/S) - \mu_y + \sigma_y\sqrt{n}\sqrt{\theta/(1-\theta)}}{\sigma_y\sqrt{n\theta(1-\theta)}}\right] - np}{\sqrt{np(1-p)}} \\ &\approx \frac{\log(K/S) - \mu_y}{\sigma_y\sqrt{p(1-p)/\theta(1-\theta)}} + \frac{\sqrt{n}(\theta - p)}{\sqrt{p(1-p)}} \end{aligned} \tag{16}$$

We will obtain

$$\begin{aligned} p' &= \frac{(R - H^-)H^+}{R(H^+ - H^-)} \\ &= \frac{(1 - R^{-1}H^-)H^+}{(H^+ - H^-)} \\ &= \frac{(1 - e^{-rt/n}H^-)H^+}{H^+ - H^-}, \quad \text{as } n \to \infty \end{aligned}$$

Hence, for large n,

$$p' = \frac{\left(1 - e^{-rt/n + \mu_y/n - (\sigma_y/\sqrt{n})\sqrt{\frac{\theta}{1-\theta}}}\right) e^{\mu_y/n + (\sigma_y/\sqrt{n})\sqrt{\frac{1-\theta}{\theta}}}}{e^{\mu_y/n + (\sigma_y/\sqrt{n})\sqrt{\frac{1-\theta}{\theta}}} - e^{\mu_y/n - (\sigma_y/\sqrt{n})\sqrt{\frac{\theta}{1-\theta}}}}$$

Let $o(1/n^a)$ denote a function tending to zero more rapidly than $1/n^a$ for all $a > 0$. By the Taylor's expansion,

$$
\begin{aligned}
H^+ &= e^{\mu_y/n+(\sigma_y/\sqrt{n})\sqrt{\frac{1-\theta}{\theta}}} \\
&= 1 + \left(\frac{\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{1-\theta}{\theta}}\right) + \frac{1}{2!}\left(\frac{\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{1-\theta}{\theta}}\right)^2 \\
&\quad + \frac{1}{3!}\left(\frac{\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{1-\theta}{\theta}}\right)^3 + \cdots \\
&= 1 + \frac{\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{1-\theta}{\theta}} + \frac{\sigma_y^2}{2n}\left(\frac{1-\theta}{\theta}\right) + o\left(\frac{1}{n}\right) \qquad (17)
\end{aligned}
$$

And

$$
\begin{aligned}
H^- &= e^{\mu_y/n-(\sigma_y/\sqrt{n})\sqrt{\frac{\theta}{1-\theta}}} \\
&= 1 + \left(\frac{\mu_y}{n} - \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}}\right) + \frac{1}{2!}\left(\frac{\mu_y}{n} - \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}}\right)^2 \\
&\quad + \frac{1}{3!}\left(\frac{\mu_y}{n} - \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}}\right)^3 + \cdots \\
&= 1 + \frac{\mu_y}{n} - \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}} + \frac{\sigma_y^2}{2n}\left(\frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n}\right) \qquad (18)
\end{aligned}
$$

Hence, the denominator of p' is

$$
\begin{aligned}
&H^+ - H^- \\
&= \frac{\sigma_y}{\sqrt{n}}\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) + \frac{\sigma_y^2}{2n}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n}\right) \\
&= \frac{\sigma_y}{\sqrt{n}}\left(\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) + \frac{\sigma_y}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)\right) \qquad (19)
\end{aligned}
$$

Meanwhile, the numerator of p' is

$$
\begin{aligned}
&(1 - e^{-rt/n} H^{-}) H^{+} \\
&= \left(1 - e^{-rt/n + \mu_y/n - (\sigma_y/\sqrt{n})\sqrt{\frac{\theta}{1-\theta}}}\right) e^{\mu_y/n + (\sigma_y/\sqrt{n})\sqrt{\frac{1-\theta}{\theta}}} \\
&= e^{\mu_y/n + (\sigma_y/\sqrt{n})\sqrt{\frac{1-\theta}{\theta}}} - e^{-rt/n + 2\mu_y/n + (\sigma_y/\sqrt{n})\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)} \\
&= H^{+} - e^{-rt/n + 2\mu_y/n + (\sigma_y/\sqrt{n})\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)} \qquad (20)
\end{aligned}
$$

Where H^{+} is given in (17).

Furthermore,

$$
\begin{aligned}
&e^{-rt/n + 2\mu_y/n + (\sigma_y/\sqrt{n})\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)} \\
&= 1 + \left[-\frac{rt}{n} + \frac{2\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)\right] \\
&\quad + \frac{1}{2!}\left[-\frac{rt}{n} + \frac{2\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)\right]^2 \\
&\quad + \frac{1}{3!}\left[-rt/n + \frac{2\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)\right]^3 + \cdots \\
&= 1 - \frac{rt}{n} + \frac{2\mu_y}{n} + \frac{\sigma_y}{\sqrt{n}}\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right) \\
&\quad + \frac{\sigma_y^2}{2n}\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)^2 + o\left(\frac{1}{n}\right) \qquad (21)
\end{aligned}
$$

Thus, the numerator of p' is

$$
\begin{aligned}
&(1 - e^{-rt/n} H^{-}) H^{+} \\
&= e^{\mu_y/n + (\sigma_y/\sqrt{n})\sqrt{\frac{1-\theta}{\theta}}} - e^{-rt/n + 2\mu_y/n + (\sigma_y/\sqrt{n})\left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}}\right)} \\
&= \frac{\sigma_y}{\sqrt{n}}\sqrt{\frac{\theta}{1-\theta}} + \frac{1}{n}\left(rt - \mu_y - \frac{1}{2}\sigma_y^2\left(\frac{\theta}{1-\theta}\right) + \sigma_y^2\right) + o\left(\frac{1}{n}\right) \qquad (22)
\end{aligned}
$$

We obtain

$$p' \approx \frac{1}{\sqrt{n}} \frac{rt - \mu_y - \frac{1}{2}\sigma_y^2\left(\frac{\theta}{1-\theta}\right) + \sigma_y^2}{\sigma_y\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) + \frac{\sigma_y^2}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)}$$

$$+ \frac{\sqrt{\frac{1-\theta}{\theta}}}{\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}} + \frac{\sigma_y}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)} + o\left(\frac{1}{n}\right) \qquad (23)$$

It can be shown that

$$\lim_{n\to\infty} p' = \frac{\sqrt{\frac{\theta}{1-\theta}}}{\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}} = \frac{\sqrt{\frac{\theta}{1-\theta}}\sqrt{\frac{1-\theta}{\theta}}}{\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right)\sqrt{\frac{1-\theta}{\theta}}}$$

$$= \frac{1}{1 + \frac{1-\theta}{\theta}} = \theta$$

Similar to the limiting result for the binomial option model of CRR, we assume that $\mu_y \to \mu t$ and $\sigma_y^2 \to \sigma^2 t$, as $n \to \infty$. Then

$$\lim_{n\to\infty} \sqrt{n}\,(\theta - p')$$

$$= -\lim_{n\to\infty} \sqrt{n}\,(p' - \theta)$$

$$= -\lim_{n\to\infty}\left(\frac{\left(rt - \mu_y - \frac{1}{2}\sigma_y^2\left(\frac{\theta}{1-\theta}\right) + \sigma_y^2\right)\sqrt{\frac{1-\theta}{\theta}}}{\sigma_y\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) + \frac{\sigma_y^2}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)}\right.$$

$$\left. + \frac{\sqrt{n}\sqrt{\frac{\theta}{1-\theta}}}{\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}} + \frac{\sigma_y}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)} - \sqrt{n}\theta\right)$$

$$= -\frac{\left(rt - \mu t - \frac{1}{2}\sigma^2 t\left(\frac{\theta}{1-\theta}\right) + \sigma^2 t\right)\sqrt{\frac{1-\theta}{\theta}}}{\sigma\sqrt{t}\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right)\sqrt{\frac{1-\theta}{\theta}}}$$

$$- \lim_{n\to\infty} \frac{\sqrt{n}\sigma\sqrt{t}\sqrt{\frac{\theta}{1-\theta}} - \sqrt{n}\sigma\sqrt{t}\theta\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) - \frac{\sigma^2 t}{2}\theta\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right)}{\sigma\sqrt{t}\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) + \frac{\sigma^2 t}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)}$$

$$= -\frac{\sqrt{\theta(1-\theta)}}{\sigma\sqrt{t}}\left(rt - \mu t - \frac{1}{2}\sigma^2 t\left(\frac{\theta}{1-\theta}\right) + \sigma^2 t - \frac{1}{2}\sigma^2 t\left(\frac{1-2\theta}{1-\theta}\right)\right)$$

$$= -\frac{\sqrt{\theta(1-\theta)}}{\sigma\sqrt{t}}\left(rt - \mu t + \frac{1}{2}\sigma^2 t\right) \tag{24}$$

From (15),

$$\lim_{n\to\infty} Z_1 = \lim_{n\to\infty}\left(\frac{\log(K/S) - \mu_y}{\sigma_y\sqrt{p'(1-p')/\theta(1-\theta)}} + \frac{\sqrt{n}(\theta - p')}{\sqrt{p'(1-p')}}\right)$$

$$= \frac{\log(K/S) - \mu t}{\sigma\sqrt{t}} - \frac{rt - \mu t + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}$$

$$= \frac{\log(K/S) - \left(r + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} \tag{25}$$

With the same manipulation on p', we have

$$p = \frac{R - H^-}{H^+ - H^-} = \frac{e^{rt/n} - H^-}{H^+ - H^-}$$

By the Taylor's series expansion, the numerator of p is

$$e^{rt/n} - H^- = \frac{1}{n}\left(rt - \mu_y - \frac{1}{2}\sigma_y^2\left(\frac{\theta}{1-\theta}\right) + \sigma_y\sqrt{n}\sqrt{\frac{\theta}{1-\theta}}\right) + o\left(\frac{1}{n}\right)$$

which is equal to $-\frac{1}{n}\sigma_y^2$ plus the numerator of p'. So that,

$$p \approx \frac{1}{\sqrt{n}}\frac{rt - \mu_y - \frac{1}{2}\sigma_y^2\left(\frac{\theta}{1-\theta}\right)}{\sigma_y\left(\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}}\right) + \frac{\sigma_y^2}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)}$$

$$+ \frac{\sqrt{\frac{\theta}{1-\theta}}}{\sqrt{\frac{\theta}{1-\theta}} + \sqrt{\frac{1-\theta}{\theta}} + \frac{\sigma_y}{2\sqrt{n}}\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{n^{1/2}}\right)} + o(1/n)$$

Hence, arguments similar to p', we have

$$\lim_{n\to\infty} p = \theta$$

And

$$\lim_{n\to\infty}\sqrt{n}(\theta - p) = -\frac{\sqrt{\theta(1-\theta)}}{\sigma\sqrt{t}}\left(rt - \mu t - \frac{1}{2}\sigma^2 t\right)$$

Hence, from (16) we have

$$\lim_{n\to\infty} Z_2 = \frac{\log(K/S) - \mu t}{\sigma\sqrt{t}} - \frac{rt - \mu t - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} = \frac{\log(K/S) - \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

Recognizing that $1 - N(Z) = N(-Z)$ and letting

$$d_1 = -\lim_{n\to\infty} Z_1$$

$$d_2 = -\lim_{n\to\infty} Z_2$$

We have the continuous time version of the two-state model

$$C = SN(d_1) - Ke^{-rt}N(d_2) \tag{26}$$

Where

$$d_1 = \frac{\log(S/K) + \left(r + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

The option pricing formula C as given in Eq. (26) is exactly that of Black and Scholes for the fixed length t of calendar time to expiration.

We close this section by noting that the advantage of the limiting result for the binomial option model of Rendleman and Barter (1979) is that no assumptions are made regarding H^+ and H^-. While in the CRR model, the corresponding parameters u, d and p are given in (8). Hence, in a way, CRR is more restricted than RB in their treatments of the convergence to the Black–Scholes formula. The only requirement in the RB paper is the understanding of the Taylor series expansion, although the convergence result in the original RB paper is not easy to follow. Thus, for a reader interested in the limiting result of the binomial option pricing model, RB seems to be a more intuitive paper and easier to comprehend without invoking advanced probability theory.

4. Multinomial Option Pricing Model

Instead of two possible movements for the stock price, as considered by CRR (1979) and RB (1979), it is natural to extend it to the situation in which there are $k + 1$ possible price movements. In this section we will present the extension proposed by Madan, Milne, and Shefrin (1989).

More details than the original paper are provided and should be helpful to the readers. We will derive the multinomial option pricing model in Sec. 4.1 and the Black and Scholes model as a limiting case in Sec. 4.2.

4.1. Derivation of the option pricing model

Suppose that the stock price follows a multiplicative multinomial process over discrete periods. The rate of return on the stock over each period can have $(k+1)$ possible values: $f_i - 1$ with probability q_i for $i = i, \ldots, (k+1)$. Thus, if the current stock price is S, the stock price at the end of the period will be one of $f_i S$. We can represent this movement with the following diagram:

$$S \to \begin{cases} f_1 S & \text{with probability} \quad q_1 \\ f_2 S & \text{with probability} \quad q_2 \\ \vdots & \qquad \vdots \qquad\qquad \vdots \\ f_{k+1} S & \text{with probability} \quad q_{k+1} \end{cases}$$

Let $X = (X_1, \ldots, X_{k+1})$ and $q = (q_1, \ldots, q_{k+1}), 0 < q_j < 1$ for $j = 1, \ldots, k+1$. X is said to have the multinomial distribution, or $X \sim \text{Mult}(n, q)$, if the probability density is

$$f(x) = \frac{n!}{x_1!x_2! \cdots x_{k+1}!} \prod_{j=1}^{k+1} q_j^{x_j}$$

For all $0 \le x_j \le n$, where $\sum_{j=1}^{k+1} x_j = n$ and $\sum_{j=1}^{k+1} q_j = 1$.

Theorem 2. *The current value of the n-period multinomial option price C is given by*

$$C = \sum_{x \in A} \frac{n!}{x_1!x_2! \cdots x_{k+1}!} \left[S \prod_{j=1}^{k+1} \left(\frac{f_j q_j}{R} \right)^{x_j} - KR^{-n} \prod_{j=1}^{k+1} q_j \right] \tag{27}$$

Where $A = \{x \mid S^* > K\} = \left\{x \mid \prod_{j=1}^{k+1} f_j^{x_j} S > K\right\}$.

Proof.

$$
\begin{aligned}
C &= R^{-n}\mathrm{E}[\max(S^* - K,\ 0)] \\
&= R^{-n}\left\{\sum_{x\in A}\frac{n!}{x_1!\,x_2!\cdots x_{k+1}!}(S^* - K)\prod_{j=1}^{k+1} q_j^{x_j}\right\} \\
&= R^{-n}\left\{\sum_{x\in A}\frac{n!}{x_1!\,x_2!\cdots x_{k+1}!}\left(S\prod_{j=1}^{k+1} f_j^{x_j} - K\right)\prod_{j=1}^{k+1} q_j^{x_j}\right\} \\
&= R^{-n}\left\{\sum_{x\in A}\frac{n!}{x_1!\,x_2!\cdots x_{k+1}!}\left(S\prod_{j=1}^{k+1}(f_j q_j)^{x_j} - K\prod_{j=1}^{k+1} q_j^{x_j}\right)\right\} \\
&= \left\{\sum_{x\in A}\frac{n!}{x_1!\,x_2!\cdots x_{k+1}!}\left(S\prod_{j=1}^{k+1}\left(\frac{f_j q_j}{R}\right)^{x_j} - KR^{-n}\prod_{j=1}^{k+1} q_j^{x_j}\right)\right\}
\end{aligned}
$$

Now, let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{k+1})^{\mathsf{T}}$ with $\nu_j = (f_j q_j)/R$. If investors are risk-neutral, that is

$$\sum_{j=1}^{k+1}(f_j S)q_j = RS$$

Implying that

$$\sum_{j=1}^{k}\frac{f_j q_j}{R} = 1$$

Hence, $\boldsymbol{\nu}$ and $\boldsymbol{q}$ are both the probability vectors in multinomial distribution. So (27) can be rewritten as

$$C = SP_\nu(A_\nu) - KR^{-n}P_q(A_q)$$

Where

$$P_\nu(A_\nu) = \sum_{x\in A}\frac{n!}{x_1!\,x_2!\cdots x_{k+1}!}\prod_{j=1}^{k+1}\nu_j^{x_j}$$

And

$$P_q(A_q) = \sum_{x\in A}\frac{n!}{x_1!\,x_2!\cdots x_{k+1}!}\prod_{j=1}^{k+1}q_j^{x_j}$$

It is noted that when $k = 1$, $q_1 = p$, $q_2 = 1 - p$, $f_1 = u$ and $f_2 = d$, then Eq. (27) is exactly the binomial option pricing formula of CRR as given in Eq. (1). Thus, Theorem 2 is a direct extension of the option pricing formula of CRR.

4.2. *The limiting case*

Denote $\eta^\top = (\eta_1, \ldots, \eta_{k+1}) = (\log(f_1), \ldots, \log(f_{k+1}))$, where $\eta_j = h\xi_j$ and

$$h = \sqrt{\frac{t}{n}}\sigma + o\left(1/\sqrt{n}\right), \quad \sum_{j=1}^{k+1} q_j \xi_j^2 = 1$$

Then

$$\log\left(\frac{S^*}{S}\right) = \sum_{j=1}^{k+1} x_j \log(f_j) = \eta^\top \boldsymbol{x}$$

Let $\eta^\top(n\boldsymbol{q}) = n\eta^\top \boldsymbol{q} \equiv \mu t$, then $\eta^\top \boldsymbol{q} = \mu t/n$, and $\left[\sum_{j=1}^{k+1} q_j \eta_j^2\right] n = nh^2 \sum_{j=1}^{k+1} q_j \xi_j^2 \equiv \sigma^2 t + \frac{\mu^2 t^2}{n}$.

Since

$$\eta^\top \boldsymbol{q} = h\xi^\top \boldsymbol{q} = \mu \frac{t}{n}, \quad h = O\left(\frac{1}{\sqrt{n}}\right)$$

We have $\xi^\top \boldsymbol{q} = O(1/\sqrt{n})$.

For any vector y and for notational convenience, let $\hat{y}$ denote the truncated vector obtain from y by deleting its last entry. That is, $y = (y_1, y_2, \ldots, y_{p-1}, y_p)^\top = (\hat{y}^\top, y_p)^\top$, where $\hat{y} = (y_1, y_2, \ldots, y_{p-1})^\top$.

Bhattacharya and Rao (1976) showed that if $\boldsymbol{X} \sim \text{Mult}(n, \boldsymbol{q})$, then for large n, $\hat{\boldsymbol{X}} \sim N_k(n\hat{\boldsymbol{q}}, n\boldsymbol{\Sigma}_q^*)$, where $\boldsymbol{\Sigma}_q^* = \Delta(\hat{\boldsymbol{q}}) - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}^\top$, and $\hat{\boldsymbol{q}}$ is a diagonal matrix such that

$$\Delta(\hat{\boldsymbol{q}}) = [b_{ij}], \quad b_{ij} = \begin{cases} q_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We next observe that

$$\hat{\boldsymbol{X}} \in A_q \quad \text{iff} \quad f_1^{x_1} f_2^{x_2} \cdots f_{k+1}^{x_{k+1}} S > K$$

Equivalently,

$$\hat{X} \in A_q \quad \text{iff} \quad \sum_{j=1}^{k+1} \log(f_j)x_j = \hat{\eta}^\top \hat{X} + \eta_{k+1}\, x_{k+1} > \log\left(\frac{K}{S}\right)$$

or

$$\hat{X} \in A_q \quad \text{iff} \quad Z^* = \frac{\hat{\eta}^\top X - n\eta^\top \hat{q}}{\left[n\hat{\eta}^\top \Sigma_q^* \hat{\eta}\right]^{1/2}} > -\frac{\log(S/K) + \eta_{k+1}x_{k+1} + n\hat{\eta}^\top \hat{q}}{\left[n\hat{\eta}^\top \Sigma_q^* \hat{\eta}\right]^{1/2}}$$

Where Z^* is a standard normal random variable and right-hand side is also a random variable since it is a function of x_{k+1}.

Next, consider $(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top(\hat{X} - n\hat{q})$ in order to obtain the limiting result, where $\mathbf{1}_k$ is a vector of unit entries and k is its dimension. Since

$$\begin{aligned}
&(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top(\hat{X} - n\hat{q}) \\
&= \hat{\eta}^\top \hat{X} - n\hat{\eta}^\top \hat{q} - \eta_{k+1}\mathbf{1}_k^\top \hat{X} + n\eta_{k+1}\mathbf{1}_k^\top \hat{q} \\
&= \hat{\eta}^\top \hat{X} - n\hat{\eta}^\top \hat{q} - \eta_{k+1}(n - x_{k+1}) + n\eta_{k+1}(1 - q_{k+1}) \\
&= \hat{\eta}^\top \hat{X} - n\hat{\eta}^\top \hat{q} + n\eta_{k+1}x_{k+1} - n\eta_{k+1}q_{k+1}
\end{aligned}$$

We have

$$\hat{\eta}^\top \hat{X} = (\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top(\hat{X} - n\hat{q}) + n\eta_{k+1}q_{k+1} - \eta_{k+1}x_{k+1}$$

Hence

$$\hat{X} \in A_q \quad \text{iff} \quad \hat{\eta}^\top \hat{X} - n\hat{\eta}^\top \hat{q} > -\log(S/K) - \eta_{k+1}x_{k+1} - n\hat{\eta}^\top \hat{q}$$

Or equivalently,

$$\text{iff} \quad (\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top(\hat{X} - n\hat{q}) > -\log(S/K) - n\eta^\top q$$

Thus,

$$P_q(A_q) = \Pr\left((\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top(\hat{X} - n\hat{q}) > -\log(S/K) - n\eta^\top q\right)$$

Consequently,

$$(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top(\hat{X} - n\hat{q}) \xrightarrow{d} N(0, n(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top \Sigma_q^* (\hat{\eta} - \eta_{k+1}\mathbf{1}_k))$$

Lemma 2. *(a) If* $\Sigma_q = \Delta(q) - qq^\top$ *then*

$$(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top \Sigma_q^* (\hat{\eta} - \eta_{k+1}\mathbf{1}_k) = (\eta - \eta_{k+1}\mathbf{1}_{k+1})^\top \Sigma_q (\eta - \eta_{k+1}\mathbf{1}_{k+1})$$

(b) If $\Sigma_\nu = \Delta(\nu) - \nu\nu^\top$ *then*

$$(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top \Sigma_\nu^*(\hat{\eta} - \eta_{k+1}\mathbf{1}_k) = (\eta - \eta_{k+1}\mathbf{1}_{k+1})^\top \Sigma_\nu(\eta - \eta_{k+1}\mathbf{1}_{k+1})$$

Proof. See the Appendix.

If

$$Z_q = \frac{\log(S/K) + n\eta^\top q}{[n(\eta - \eta_{k+1}1_{k+1})^\top \Sigma_q(\eta - \eta_{k+1}1_{k+1})]^{1/2}}$$

Then

$$P_q(A_q) = \Pr\left(\frac{(\hat{\eta} - \eta_{k+1}1_k)^\top(\hat{X} - n\hat{q})}{[n(\eta - \eta_{k+1}1_{k+1})^\top \Sigma_q(\eta - \eta_{k+1}1_{k+1})]^{1/2}} > -Z_q\right)$$

$$\approx 1 - N(-Z_q) = N(Z_q)$$

Similarly, if

$$Z_\nu = \frac{\log(S/K) + n\eta^\top \nu}{[n(\eta - \eta_{k+1}1_{k+1})^\top \Sigma_\nu(\eta - \eta_{k+1}1_{k+1})]^{1/2}}$$

Then $P_\nu(A_\nu) = N(Z_\nu)$.

Theorem 3. *Suppose that* $R^{-1}f^\top q = (1+r)^{-t/n}f^\top q = 1, \eta_j = \log(f_j) = h\xi_j, h = \sqrt{t/n}\sigma + o(1/\sqrt{n})$ *for all* j *and* $\sum_{j=1}^{k+1} q_j\xi_j^2 = 1$.
Then

(a) $\eta^\top q = \frac{t}{n}\left(\log(1+r) - \frac{\sigma^2}{2}\right) + o\left(\frac{1}{n}\right)$

(b) $\eta^\top \nu = \frac{t}{n}\left(\log(1+r) + \frac{\sigma^2}{2}\right) + o\left(\frac{1}{n}\right)$

(c) $\lim_{n\to\infty} n(\hat{\eta} - \eta_{k+1}1_k)^\top \Sigma_q^*(\hat{\eta} - \eta_{k+1}1_k)$
$= \lim_{n\to\infty} n(\hat{\eta} - \eta_{k+1}1_{k+1})^\top \Sigma_q(\eta - \eta_{k+1}1_{k+1}) = \sigma^2 t$

(d) $\lim_{n\to\infty} n(\hat{\eta} - \eta_{k+1}1_k)^\top \Sigma_\nu^*(\hat{\eta} - \eta_{k+1}1_k) = \sigma^2 t$

Proof. See the Appendix.

Using the results in Theorem 3, let

$$d_1 = \lim_{n\to\infty} Z_\nu = \frac{\log(S/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = \lim_{n\to\infty} Z_q = \frac{\log(S/K) + (r - \sigma^2/2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

then $C \to SN(d_1) - Ke^{-rt}N(d_2)$,which is the Black–Scholes formula. Here we use the result R^{-n} tends to e^{-rt} as $n \to \infty$, as seen in Sec. 2.2 and the fact that $P_\nu(A_\nu) \to N(d_1)$ and $P_q(A_q) \to N(d_2)$.

Appendix

Proof of Lemma 2.
(a) Since

$$\Sigma_q = \Delta(q) - qq^\top = \Delta\begin{pmatrix}\hat{q}\\ q_{k+1}\end{pmatrix} - \begin{pmatrix}\hat{q}\\ q_{k+1}\end{pmatrix}(\hat{q}^\top, q_{k+1})$$

$$= \begin{pmatrix}\Delta(\hat{q}) & 0\\ 0^\top & q_{k+1}\end{pmatrix} - \begin{pmatrix}\hat{q}\hat{q}^\top & q_{k+1}\hat{q}\\ q_{k+1}\hat{q}^\top & q_{k+1}^2\end{pmatrix}$$

$$= \begin{pmatrix}\Sigma_q^* & -q_{k+1}\hat{q}\\ -q_{k+1}\hat{q}^\top & q_{k+1} - q_{k+1}^2\end{pmatrix}$$

And

$$\eta = \begin{pmatrix}\hat{\eta}\\ \eta_{k+1}\end{pmatrix}$$

We have

$$(\eta - \eta_{k+1}\mathbf{1}_{k+1})^\top\Sigma_q(\eta - \eta_{k+1}\mathbf{1}_{k+1})$$

$$= [(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top, 0]\begin{pmatrix}\Sigma_q^* & -q_{k+1}\hat{q}\\ -q_{k+1}\hat{q}^\top & q_{k+1} - q_{k+1}^2\end{pmatrix}\begin{pmatrix}\hat{\eta} - \eta_{k+1}\mathbf{1}_k\\ 0\end{pmatrix}$$

$$= (\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top\Sigma_q^*(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)$$

(b) The proof is similar to that of (a).

Proof of Theorem 3.
(a) Since $(1+r)^{-t/n}f^\top q = 1$ and $\mathbf{1}^\top q = 1$, we have

$$((1+r)^{-t/n}f - \mathbf{1}_{k+1})^\top q = 0$$

Hence

$$(1+r)^{-t/n}f_j - 1 = (1+r)^{-t/n}e^{h\xi_j} - 1$$

$$= e^{-\frac{t}{n}\log(1+r)+h\xi_j} - 1$$

$$= \left(-\frac{t}{n}\log(1+r) + \eta_i\right) + \frac{1}{2}h^2\xi_j^2 + o\left(\frac{1}{n}\right)$$

Making use of the fact that

$$\eta_j = \log(f_j) = h\xi_j, \quad h = \sqrt{\frac{t}{n}}\sigma + o\left(\frac{1}{\sqrt{n}}\right)$$

and if $A = O(1/n)$ and $B = O(1/n)$,

$$e^{A+B} - 1 = A + B + \frac{1}{2}B^2 + o\left(\frac{1}{n}\right)$$

We have,

$$\begin{aligned}
0 &= ((1+r)^{-t/n} f - \mathbf{1}_{k+1})^\top \boldsymbol{q} \\
&= \sum_{j=1}^{k+1} ((1+r)^{-t/n} f_j - 1) q_j \\
&= \sum_{j=1}^{k+1} \left\{ \left(-\frac{t}{n}\log(1+r) + \eta_j \right) + \frac{1}{2}h^2\xi_j^2 + o\left(\frac{1}{n}\right) \right\} q_j \\
&= -\frac{t}{n}\log(1+r) + \boldsymbol{\eta}^\top \boldsymbol{q} + \frac{1}{2}h^2 \sum_{j=1}^{k+1} \xi_j^2 q_j + o\left(\frac{1}{n}\right)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\boldsymbol{\eta}^\top \boldsymbol{q} &= \frac{t}{n}\log(1+r) - \frac{1}{2}\frac{\sigma^2 t}{n} + o\left(\frac{1}{n}\right) \\
&= \frac{(\log(1+r) - \sigma^2/2)t}{n} + o\left(\frac{1}{n}\right)
\end{aligned}$$

Multiplying both sides by $\sqrt{n}$, we obtain

$$(\sqrt{n}h)\boldsymbol{\xi}^\top \boldsymbol{q} = \frac{t}{\sqrt{n}}\left(\log(1+r) - \frac{\sigma^2}{2}\right) + o\left(\frac{1}{n}\right)$$

Thus,

$$\boldsymbol{\xi}^\top \boldsymbol{q} = 0, \quad \text{as } n \to \infty$$

(b) Since

$$
\begin{aligned}
&((1+r)^{-t/n}\boldsymbol{f} - \mathbf{1}_{k+1})^{\top}\boldsymbol{\nu} \\
&= \sum_{j=1}^{k+1}((1+r)^{-t/n}f_j - 1)\nu_j \\
&= \sum_{j=1}^{k+1}\left\{\left(-\frac{t}{n}\log(1+r) + \eta_j\right) + \frac{1}{2}h^2\xi_j^2 + o\left(\frac{1}{n}\right)\right\}\nu_j \\
&= -\frac{t}{n}\log(1+r) + \boldsymbol{\eta}^{\top}\boldsymbol{\nu} + \frac{1}{2}\frac{\sigma^2 t}{n}\sum_{j=1}^{k+1}\xi_j^2\frac{f_j q_j}{R} + o\left(\frac{1}{n}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\sum_{j=1}^{k+1}\xi_j^2\frac{f_j q_j}{R} &= \sum_{j=1}^{k+1}\{((1+r)^{-t/n}f_j)q_j\xi_j^2\} \\
&= \sum_{j=1}^{k+1}[\{((1+r)^{-t/n}f_j - 1) + 1\}q_j\xi_j^2] \\
&= \sum_{j=1}^{k+1}\left\{1 - \frac{t}{n}\log(1+r) + \eta_j + \frac{1}{2}\eta_j^2 + o\left(\frac{1}{n}\right)\right\}q_j\xi_j^2 \\
&= \sum_{j=1}^{k+1}q_j\xi_j^2 + o\left(\frac{1}{n}\right)
\end{aligned}
$$

We have

$$
((1+r)^{-t/n}\boldsymbol{f} - \mathbf{1}_{k+1})^{\top}\boldsymbol{\nu} = -\frac{t}{n}\log(1+r) + \boldsymbol{\eta}^{\top}\boldsymbol{\nu} + \frac{\sigma^2 t}{2n} + o\left(\frac{1}{n}\right)
$$

On the other hand,

$$
\nu_j = (1+r)^{-t/n}f_j q_j = (1+r)^{-t/n}e^{\eta_j}q_j
$$

Implying that

$$
\begin{aligned}
&((1+r)^{-t/n} \boldsymbol{f} - \mathbf{1}_{k+1})^{\top} \boldsymbol{\nu} \\
&= \sum_{j=1}^{k+1} \left(e^{\eta_j - \frac{t}{n}\log(1+r)} - 1 \right) \left(e^{\eta_j - \frac{t}{n}\log(1+r)} \right) q_j \\
&= \sum_{j=1}^{k+1} \left(-\frac{t}{n}\log(1+r) + \eta_j + \frac{1}{2}\eta_j^2 + o\left(\frac{1}{n}\right) \right) \\
&\quad \times \left(1 - \frac{t}{n}\log(1+r) + \eta_j + \frac{1}{2}\eta_j^2 + 0\left(\frac{1}{n}\right) \right) q_j \\
&= \sum_{j=1}^{k+1} \left(\eta_j - \frac{t}{n}\log(1+r) + \frac{3}{2}\eta_j^2 \right) q_j + o\left(\frac{1}{n}\right) \\
&= \boldsymbol{\eta}^{\top}\boldsymbol{q} - \frac{t}{n}\log(1+r) + \frac{3}{2}\sum_{j=1}^{k+1} h^2 \xi_j^2 q_j + o\left(\frac{1}{n}\right) \\
&= \boldsymbol{\eta}^{\top}\boldsymbol{q} - \frac{t}{n}\log(1+r) + \frac{3}{2}\frac{\sigma^2 t}{n} + o\left(\frac{1}{n}\right) \\
&= -\frac{t}{n}\log(1+r) + \boldsymbol{\eta}^{\top}\boldsymbol{\nu} + \frac{1}{2}\frac{\sigma^2 t}{n} + o\left(\frac{1}{n}\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\boldsymbol{\eta}^{\top}\boldsymbol{\nu} &= \boldsymbol{\eta}^{\top}\boldsymbol{q} + \frac{\sigma^2 t}{n} + o\left(\frac{1}{n}\right) \\
&= \frac{(\log(1+r) + \sigma^2/2)t}{n} + o\left(\frac{1}{n}\right)
\end{aligned}
$$

(c) Let $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)^{\top}$, $\psi_j = \xi_j - \xi_{k+1}$, $j = 1, \ldots, k$. We have

$$
\hat{\boldsymbol{\eta}} - \eta_{k+1}\mathbf{1}_k = h\boldsymbol{\psi}
$$

Then

$$\begin{aligned}
&\lim_{n\to\infty} n(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^\top \boldsymbol{\Sigma}_q^*(\hat{\eta} - \eta_{k+1}\mathbf{1}_k) \\
&\quad = \lim_{n\to\infty} nh^2 \boldsymbol{\psi}^\top(\Delta(\boldsymbol{q}) - \boldsymbol{q}\boldsymbol{q}^\top)\boldsymbol{\psi} \\
&\quad = \sigma^2 t \boldsymbol{\psi}^\top(\Delta(\boldsymbol{q}) - \boldsymbol{q}\boldsymbol{q}^\top)\boldsymbol{\psi}
\end{aligned}$$

Now

$$\begin{aligned}
\boldsymbol{\psi}^\top \Delta(\hat{\boldsymbol{q}})\boldsymbol{\psi} &= \sum_{j=1}^{k} \psi_j^2 q_j \\
&= \sum_{j=1}^{k} q_j(\xi_j - \xi_{k+1})^2 \\
&= \sum_{j=1}^{k} q_j \left(\xi_j^2 + \xi_{k+1}^2 - 2\xi_j\xi_{k+1}\right) \\
&= \sum_{j=1}^{k+1} q_j\xi_j^2 - q_{k+1}\xi_{k+1}^2 + \xi_{k+1}^2(1 - q_{k+1}) \\
&\quad - 2\xi_{k+1}^2 \left(\boldsymbol{\xi}^\top \boldsymbol{q} - q_{k+1}\xi_{k+1}\right) \\
&= 1 + \xi_{k+1}^2
\end{aligned}$$

Also

$$\begin{aligned}
\boldsymbol{\psi}^\top \hat{\boldsymbol{q}} &= \sum_{j=1}^{k} q_j(\xi_j - \xi_{k+1}) \\
&= \sum_{j=1}^{k} q_j(\xi_j - \xi_{k+1}) - (\xi_j - \xi_{k+1})q_{k+1} \\
&= \boldsymbol{\xi}^\top \hat{\boldsymbol{q}} - \xi_{k+1} \\
&= -\xi_{k+1}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} n(\eta - \eta_{k+1}\mathbf{1}_{k+1})^{\top}\boldsymbol{\Sigma}_q(\eta - \eta_{k+1}\mathbf{1}_{k+1}) \\
&\quad = \lim_{n\to\infty} n(\hat{\eta} - \eta_{k+1}\mathbf{1}_k)^{\top}\boldsymbol{\Sigma}_q^*(\hat{\eta} - \eta_{k+1}\mathbf{1}_k) \\
&\quad = \sigma^2 t[1 + \xi_{k+1}^2 - (-\xi_{k+1})^2] \\
&\quad = \sigma^2 t
\end{aligned}$$

(d) The proof is analogous to that of (c) and hence is omitted.

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