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Supplementary Material for "Extending multivariate-t linear mixed models for multiple longitudinal data with censored responses and heavy tails"

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This supporting information is a longer version of the printed paper. It contains the detailed proofs of Propositions 2 and 3, and the missing information matrix of $\boldsymbol{\beta}$ together with Supplementary Table for the simulation study.

Web Appendix A: Proof of Proposition 2.

Using the fact of $\boldsymbol{y}_i = \boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}} + \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{c}}$, it follows from Proposition 3 of Matos et al. (2013) and Proposition 1 of this paper that

$$\begin{aligned}
\widehat{\tau\boldsymbol{y}}_i^{(h)} &= E\big[\tau_i\boldsymbol{y}_i|\boldsymbol{u}_i,\boldsymbol{c}_i,\hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big](\boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}} + \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{c}})\big|\boldsymbol{u}_i,\boldsymbol{c}_i,\hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= \boldsymbol{O}_i^{\mathrm{T}}E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\Big|\boldsymbol{u}_i,\boldsymbol{c}_i,\hat{\boldsymbol{\theta}}^{(h)}\Big]\boldsymbol{y}_i^{\mathrm{o}} + \boldsymbol{C}_i^{\mathrm{T}}E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i^{\mathrm{c}}\Big|\boldsymbol{u}_i,\boldsymbol{c}_i,\hat{\boldsymbol{\theta}}^{(h)}\Big] \\
&= \hat{\tau}_i^{(h)}\big(\boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}} + \boldsymbol{C}_i^{\mathrm{T}}E[\boldsymbol{W}_i^{\mathrm{c}}]\big),
\end{aligned}$$

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and

$$\begin{aligned}
\widehat{\tau \boldsymbol{y}_i^2}^{(h)} &= E\big[\tau_i \boldsymbol{y}_i \boldsymbol{y}_i^{\mathrm{T}} | \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big](\boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}} + \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{c}})(\boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}} + \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{c}})^{\mathrm{T}}\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}}\boldsymbol{y}_i^{\mathrm{o}^{\mathrm{T}}}\boldsymbol{O}_i + \boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}} E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i^{\mathrm{c}^{\mathrm{T}}}\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\boldsymbol{C}_i \\
&\quad + \boldsymbol{C}_i^{\mathrm{T}} E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i^{\mathrm{c}}\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\boldsymbol{y}_i^{\mathrm{o}^{\mathrm{T}}}\boldsymbol{O}_i + \boldsymbol{C}_i^{\mathrm{T}} E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i^{\mathrm{c}}\boldsymbol{y}_i^{\mathrm{c}^{\mathrm{T}}}\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\boldsymbol{C}_i \\
&= \hat{\tau}_i^{(h)}\Big\{\boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}}\boldsymbol{y}_i^{\mathrm{o}^{\mathrm{T}}}\boldsymbol{O}_i + \boldsymbol{O}_i^{\mathrm{T}}\boldsymbol{y}_i^{\mathrm{o}}E^{\mathrm{T}}[\boldsymbol{W}_i^{\mathrm{c}}]\boldsymbol{C}_i + \boldsymbol{C}_i^{\mathrm{T}}E[\boldsymbol{W}_i^{\mathrm{c}}]\boldsymbol{y}_i^{\mathrm{o}^{\mathrm{T}}}\boldsymbol{O}_i + \boldsymbol{C}_i^{\mathrm{T}}E[\boldsymbol{W}_i^{\mathrm{c}}\boldsymbol{W}_i^{\mathrm{c}^{\mathrm{T}}}]\boldsymbol{C}_i\Big\},
\end{aligned}$$

where the detailed derivations of $\hat{\tau}_i^{(h)}$ and $\boldsymbol{W}_i^{\mathrm{c}}$ are given in the proof of Web Appendix B. Note that throughout the supplementary material, we use the notation, e.g., $E_X[f(X, \ldots)|y, z, \ldots]$, to indicate that the term inside $f(X, \ldots)$ needed to be integrated over the conditional distribution X given $y, z, \ldots$ is only X under this conditional expectation.

According to the hierarchical formulation of MtLMMC given in (4), we use the standard matrix factorizations (cf. Anderson (2003), Appendix A.3) to obtain

$$\boldsymbol{b}_i|\boldsymbol{y}_i, \tau_i \sim \mathcal{N}_q\big(\boldsymbol{\Sigma}_{\boldsymbol{b}_i}\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{R}_i^{-1}(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta}), \tau_i^{-1}\boldsymbol{\Sigma}_{\boldsymbol{b}_i}\big).$$

This conditional distribution is useful for evaluating the desired conditional moments of latent data, including

$$\begin{aligned}
\widehat{\tau \boldsymbol{b}_i}^{(h)} &= E\big[\tau_i \boldsymbol{b}_i | \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i E_{\boldsymbol{b}_i}\big(\boldsymbol{b}_i|\boldsymbol{y}_i, \tau_i\big)\big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i\boldsymbol{\Sigma}_{\boldsymbol{b}_i}\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{R}_i^{-1}(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})\big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[\boldsymbol{\Sigma}_{\boldsymbol{b}_i}\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{R}_i^{-1}E_{\tau_i}\big[\tau_i(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\Big[\boldsymbol{\Sigma}_{\boldsymbol{b}_i}\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{R}_i^{-1}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})\Big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big] \\
&= \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)}\boldsymbol{Z}_i^{\mathrm{T}}\hat{\boldsymbol{R}}_i^{(h)^{-1}}(\widehat{\tau\boldsymbol{y}}_i^{(h)} - \hat{\tau}_i^{(h)}\boldsymbol{X}_i\hat{\boldsymbol{\beta}}^{(h)}),
\end{aligned}$$

$$\begin{aligned}
\widehat{\tau \boldsymbol{b}_i^2}^{(h)} &= E\big[\tau_i \boldsymbol{b}_i \boldsymbol{b}_i^{\mathrm{T}} | \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i E_{\boldsymbol{b}_i}\big(\boldsymbol{b}_i \boldsymbol{b}_i^{\mathrm{T}} | \boldsymbol{y}_i, \tau_i\big)\big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i\big(\boldsymbol{\Sigma}_{\boldsymbol{b}_i} \boldsymbol{Z}_i^{\mathrm{T}} \boldsymbol{R}_i^{-1}(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}} \boldsymbol{R}_i^{-1} \boldsymbol{Z}_i \boldsymbol{\Sigma}_{\boldsymbol{b}_i} + \tau_i^{-1}\boldsymbol{\Sigma}_{\boldsymbol{b}_i}\big)|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} \boldsymbol{Z}_i^{\mathrm{T}} \hat{\boldsymbol{R}}_i^{(h)^{-1}} E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \hat{\boldsymbol{R}}_i^{(h)^{-1}} \boldsymbol{Z}_i \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} + \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} \\
&= \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} \boldsymbol{Z}_i^{\mathrm{T}} \hat{\boldsymbol{R}}_i^{(h)^{-1}} E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)(\boldsymbol{y}_i \boldsymbol{y}_i^{\mathrm{T}} - \boldsymbol{y}_i \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i^{\mathrm{T}} - \boldsymbol{X}_i \boldsymbol{\beta} \boldsymbol{y}_i^{\mathrm{T}} + \boldsymbol{X}_i \boldsymbol{\beta} \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i^{\mathrm{T}})\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big] \\
&\quad \times \hat{\boldsymbol{R}}_i^{(h)^{-1}} \boldsymbol{Z}_i \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} + \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} \\
&= \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} \boldsymbol{Z}_i^{\mathrm{T}} \hat{\boldsymbol{R}}_i^{(h)^{-1}} \Big[\widehat{\tau \boldsymbol{y}_i^2}^{(h)} - \widehat{\tau \boldsymbol{y}_i}^{(h)} \hat{\boldsymbol{\beta}}^{(h)^{\mathrm{T}}} \boldsymbol{X}_i^{\mathrm{T}} - \boldsymbol{X}_i \boldsymbol{\beta}^{(h)} \widehat{\tau \boldsymbol{y}_i}^{(h)^{\mathrm{T}}} + \hat{\tau}_i^{(h)} \boldsymbol{X}_i \hat{\boldsymbol{\beta}}^{(h)} \hat{\boldsymbol{\beta}}^{(h)^{\mathrm{T}}} \boldsymbol{X}_i^{\mathrm{T}}\Big] \\
&\quad \times \hat{\boldsymbol{R}}_i^{(h)^{-1}} \boldsymbol{Z}_i \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} + \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)},
\end{aligned}$$

$$\begin{aligned}
\widehat{\tau \boldsymbol{y} \boldsymbol{b}_i}^{(h)} &= E\big[\tau_i \boldsymbol{y}_i \boldsymbol{b}_i^{\mathrm{T}} \big| \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i \boldsymbol{y}_i E_{\boldsymbol{b}_i}\big(\boldsymbol{b}_i^{\mathrm{T}} | \boldsymbol{y}_i, \tau_i\big)\big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i \boldsymbol{y}_i (\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}} \boldsymbol{R}_i^{-1} \boldsymbol{Z}_i \boldsymbol{\Sigma}_{\boldsymbol{b}_i} \big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\big[\boldsymbol{y}_i(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big] \hat{\boldsymbol{R}}_i^{(h)^{-1}} \boldsymbol{Z}_i \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)} \\
&= (\widehat{\tau \boldsymbol{y}_i^2}^{(h)} - \widehat{\tau \boldsymbol{y}_i}^{(h)} \hat{\boldsymbol{\beta}}^{(h)^{\mathrm{T}}} \boldsymbol{X}_i^{\mathrm{T}}) \hat{\boldsymbol{R}}_i^{(h)^{-1}} \boldsymbol{Z}_i \hat{\boldsymbol{\Sigma}}_{\boldsymbol{b}_i}^{(h)},
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\tau \boldsymbol{E}_i}^{(h)} &= E\big[\tau_i(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \boldsymbol{Z}_i\boldsymbol{b}_i)(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \boldsymbol{Z}_i\boldsymbol{b}_i)^{\mathrm{T}}\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i E_{\boldsymbol{b}_i}\big((\boldsymbol{y}_i \boldsymbol{y}_i^{\mathrm{T}} - \boldsymbol{y}_i \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i^{\mathrm{T}} - \boldsymbol{y}_i \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{Z}_i^{\mathrm{T}} - \boldsymbol{X}_i \boldsymbol{\beta} \boldsymbol{y}_i^{\mathrm{T}} - \boldsymbol{Z}_i \boldsymbol{b}_i \boldsymbol{y}_i^{\mathrm{T}} + \boldsymbol{X}_i \boldsymbol{\beta} \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i^{\mathrm{T}} \\
&\quad + \boldsymbol{Z}_i \boldsymbol{b}_i \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{Z}_i^{\mathrm{T}} + \boldsymbol{X}_i \boldsymbol{\beta} \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{Z}_i^{\mathrm{T}} + \boldsymbol{Z}_i \boldsymbol{b}_i \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i^{\mathrm{T}})|\boldsymbol{y}_i, \tau_i\big)\big|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= \widehat{\tau \boldsymbol{y}_i^2}^{(h)} - \boldsymbol{X}_i \boldsymbol{\beta}\big(\widehat{\tau \boldsymbol{y}_i}^{(h)^{\mathrm{T}}} - \widehat{\tau \boldsymbol{b}_i}^{(h)^{\mathrm{T}}} \boldsymbol{Z}_i^{\mathrm{T}}\big) - (\widehat{\tau \boldsymbol{y}_i}^{(h)} - \boldsymbol{Z}_i \widehat{\tau \boldsymbol{b}_i}^{(h)}) \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i \\
&\quad - \widehat{\tau \boldsymbol{y} \boldsymbol{b}_i}^{(h)} \boldsymbol{Z}_i^{\mathrm{T}} - \boldsymbol{Z}_i \widehat{\tau \boldsymbol{y} \boldsymbol{b}_i}^{(h)} + \hat{\tau}_i^{(h)} \boldsymbol{X}_i \boldsymbol{\beta} \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}_i^{\mathrm{T}} + \boldsymbol{Z}_i \widehat{\tau \boldsymbol{b}_i^2}^{(h)} \boldsymbol{Z}_i^{\mathrm{T}}.
\end{aligned}$$

This completes the proof of Proposition 2. □

Web Appendix B: Proof of Proposition 3.

From hierarchy (6), dividing the joint PDF of $(\boldsymbol{y}_i, \tau_i)$ by the marginal PDF of $\boldsymbol{y}_i$ yields

$$\tau_i | \boldsymbol{y}_i \sim \mathrm{Gamma}\Big(\frac{\nu + n_i}{2}, \frac{\nu + \Delta_{\boldsymbol{y}_i}}{2}\Big),$$

where $\Delta_{\boldsymbol{y}_i} = (\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta})$. Using this fact, we can derive the formulae for $\hat{\tau}_i^{(h)}$ and the distribution of $\boldsymbol{W}_i^{\mathrm{c}}$ under the following three censoring patterns, say (i) only non-censored, (ii) only censored, and (iii) both censored and non-censored situations.

(i) When the ith subject has only non-censored (observed) measurements, we have $\boldsymbol{u}_i = \boldsymbol{y}_i$, and thus

$$\begin{aligned}
\hat{\tau}_i^{(h)} &= E\big[\tau_i|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] = E\big[\tau_i|\boldsymbol{y}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] = (\hat{\nu}^{(h)} + n_i)\big/(\hat{\nu}^{(h)} + \hat{\Delta}_{\boldsymbol{y}_i}^{(h)}),\\
\widehat{\tau\boldsymbol{y}}_i^{(h)} &= E\big[\tau_i\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] = E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\boldsymbol{y}_i = \hat{\tau}_i^{(h)}\boldsymbol{y}_i,\\
\widehat{\tau\boldsymbol{y}_i^2}^{(h)} &= E\big[\tau_i\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] = E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}} = \hat{\tau}_i^{(h)}\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}.
\end{aligned}$$

(ii) When the ith subject has only censored measurements, the auxiliary permutation matrix $\boldsymbol{O}_i$ is a null set and $\boldsymbol{C}_i = \boldsymbol{I}_{n_i}$. Thus $\boldsymbol{y}_i = \boldsymbol{y}_i^{\mathrm{c}}$, and $\boldsymbol{y}_i \sim \mathcal{T}t_{n_i}(\boldsymbol{X}_i\boldsymbol{\beta}, \frac{\nu}{\nu+2}\boldsymbol{\Lambda}_i, \nu+2; \mathbb{A}_i)$, where $\mathbb{A}_i = \{\boldsymbol{y}_i = [y_{ijk}] \mid y_{ijk} \le u_{ijk}, j = 1, \ldots, r, k = 1, \ldots, s_i\}$. Accordingly,

$$\begin{aligned}
\hat{\tau}_i^{(h)} &= E\big[\tau_i|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big]\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\\
&= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\Big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\\
&= \frac{\mathcal{T}_{n_i}\big(\boldsymbol{u}_i|\boldsymbol{X}_i\hat{\boldsymbol{\beta}}^{(h)}, \frac{\hat{\nu}^{(h)}}{\hat{\nu}^{(h)}+2}\hat{\boldsymbol{\Lambda}}_i^{(h)}, \hat{\nu}^{(h)} + 2\big)}{\mathcal{T}_{n_i}\big(\boldsymbol{u}_i|\boldsymbol{X}_i\hat{\boldsymbol{\beta}}^{(h)}, \hat{\boldsymbol{\Lambda}}_i^{(h)}, \hat{\nu}^{(h)}\big)},
\end{aligned} \tag{S.1}$$

in which the last equality follows from Proposition 2 of Matos et al. (2013) with $k = 0$. Furthermore, by Proposition 2 of Matos et al. (2013) with $k = 1$ and 2, we obtain

$$\begin{aligned}
\widehat{\tau\boldsymbol{y}}_i^{(h)} &= E\big[\tau_i\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big]\boldsymbol{y}_i\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\\
&= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i\Big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\\
&= \hat{\tau}_i^{(h)}E[\boldsymbol{W}_i^{\mathrm{c}}],
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\tau\boldsymbol{y}_i^2}^{(h)} &= E\big[\tau_i\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\\
&= E_{\boldsymbol{y}_i}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big]\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}\big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big]\\
&= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu + n_i}{\nu + \Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}\Big|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big]\\
&= \hat{\tau}_i^{(h)}E[\boldsymbol{W}_i^{\mathrm{c}}\boldsymbol{W}_i^{\mathrm{c}\,\mathrm{T}}],
\end{aligned}$$

where $\hat{\tau}_i^{(h)}$ is given by (S.1), and $\boldsymbol{W}_i^{\mathrm{c}} \sim \mathcal{T}t_{n_i}(\boldsymbol{X}_i\hat{\boldsymbol{\beta}}^{(h)}, \frac{\hat{\nu}^{(h)}}{\hat{\nu}^{(h)}+2}\hat{\boldsymbol{\Lambda}}_i^{(h)}, \hat{\nu}^{(h)}+2; \mathbb{A}_i)$ with $\hat{\boldsymbol{\Lambda}}_i^{(h)}$ being $\boldsymbol{\Lambda}_i$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}^{(h)}$.

(iii) When the ith subject has both censored and non-censored measurements, using the fact that $\{\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{c}_i\}$, $\{\boldsymbol{y}_i|\boldsymbol{u}_i, \boldsymbol{c}_i, \boldsymbol{y}_i^{\mathrm{o}}\}$, and $\{\boldsymbol{y}_i^{\mathrm{c}}|\boldsymbol{u}_i, \boldsymbol{c}_i, \boldsymbol{y}_i^{\mathrm{o}}\}$ are equivalent processes, we have

$$\begin{aligned}
\hat{\tau}_i^{(h)} &= E\big[\tau_i|\boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i^{\mathrm{c}}}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big]\big|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i^{\mathrm{c}}}\bigg[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\bigg|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\bigg] \\
&= \bigg(\frac{\hat{\nu}^{(h)}+n_i^{\mathrm{o}}}{\hat{\nu}^{(h)}+\hat{\Delta}_{\boldsymbol{y}_i^{\mathrm{o}}}^{(h)}}\bigg)\frac{\mathcal{T}_{n_i-n_i^{\mathrm{o}}}\big(\boldsymbol{u}_i^{\mathrm{c}}|\hat{\boldsymbol{\mu}}_i^{\mathrm{c}\cdot\mathrm{o}^{(h)}}, \frac{\hat{\nu}^{(h)}+n_i^{\mathrm{o}}}{\hat{\nu}^{(h)}+n_i^{\mathrm{o}}+2}\hat{\boldsymbol{S}}_i^{\mathrm{cc}\cdot\mathrm{o}^{(h)}}, \hat{\nu}^{(h)}+n_i^{\mathrm{o}}+2\big)}{\mathcal{T}_{n_i-n_i^{\mathrm{o}}}\big(\boldsymbol{u}_i^{\mathrm{c}}|\hat{\boldsymbol{\mu}}_i^{\mathrm{c}\cdot\mathrm{o}^{(h)}}, \hat{\boldsymbol{S}}_i^{\mathrm{cc}\cdot\mathrm{o}^{(h)}}, \hat{\nu}^{(h)}+n_i^{\mathrm{o}}\big)}, \qquad \text{(S.2)}
\end{aligned}$$

in which the last equality follows from the conditional distribution of $\boldsymbol{y}_i^{\mathrm{c}}|\boldsymbol{y}_i^{\mathrm{o}}$ specified in Proposition 1 of this paper and Proposition 3 of Matos et al. (2013) with $k=0$, where $\hat{\Delta}_{\boldsymbol{y}_i^{\mathrm{o}}}^{(h)} = (\boldsymbol{y}_i^{\mathrm{o}} - \boldsymbol{X}_i^{\mathrm{o}}\hat{\boldsymbol{\beta}}^{(h)})^{\mathrm{T}}\hat{\boldsymbol{\Lambda}}_i^{\mathrm{oo}(h)^{-1}}(\boldsymbol{y}_i^{\mathrm{o}} - \boldsymbol{X}_i^{\mathrm{o}}\hat{\boldsymbol{\beta}}^{(h)})$ with $\hat{\boldsymbol{\Lambda}}_i^{\mathrm{oo}(h)}$ being $\boldsymbol{\Lambda}_i^{\mathrm{oo}}$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}^{(h)}$. Under this case, using Proposition 3 of Matos et al. (2013) again with $k=1$ and 2, the conditional moments involving the latent data τ_i and $\boldsymbol{y}_i^{\mathrm{c}}$ within $\widehat{\tau\boldsymbol{y}_i}^{(h)}$ and $\widehat{\tau\boldsymbol{y}_i^2}^{(h)}$ in Proposition 2 become

$$\begin{aligned}
E\big[\tau_i\boldsymbol{y}_i^{\mathrm{c}}|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] &= E_{\boldsymbol{y}_i^{\mathrm{c}}}\big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i^{\mathrm{c}}\big]\boldsymbol{y}_i^{\mathrm{c}}|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] \\
&= E_{\boldsymbol{y}_i^{\mathrm{c}}}\bigg[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i^{\mathrm{c}}|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\bigg] \\
&= \hat{\tau}_i^{(h)}E[\boldsymbol{W}_i^{\mathrm{c}}],
\end{aligned}$$

and

$$\begin{aligned}
E\big[\tau_i\boldsymbol{y}_i^{\mathrm{c}}\boldsymbol{y}_i^{\mathrm{c}^{\mathrm{T}}}|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\big] &= E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big[E_{\tau_i}\big[\tau_i|\boldsymbol{y}_i\big]\boldsymbol{y}_i^{\mathrm{c}}\boldsymbol{y}_i^{\mathrm{c}^{\mathrm{T}}}|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\Big] \\
&= E_{\boldsymbol{y}_i^{\mathrm{c}}}\bigg[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{y}_i^{\mathrm{c}}\boldsymbol{y}_i^{\mathrm{c}^{\mathrm{T}}}|\boldsymbol{y}_i^{\mathrm{o}}, \boldsymbol{u}_i, \boldsymbol{c}_i, \hat{\boldsymbol{\theta}}^{(h)}\bigg] \\
&= \hat{\tau}_i^{(h)}E[\boldsymbol{W}_i^{\mathrm{c}}\boldsymbol{W}_i^{\mathrm{c}^{\mathrm{T}}}],
\end{aligned}$$

where $\hat{\tau}_i^{(h)}$ is given by (S.2), and $\boldsymbol{W}_i^{\mathrm{c}} \sim \mathcal{T}t_{n_i-n_i^{\mathrm{o}}}(\hat{\boldsymbol{\mu}}_i^{\mathrm{c}\cdot\mathrm{o}^{(h)}}, \frac{\hat{\nu}^{(h)}+n_i^{\mathrm{o}}}{\hat{\nu}^{(h)}+n_i^{\mathrm{o}}+2}\hat{\boldsymbol{S}}_i^{\mathrm{cc}\cdot\mathrm{o}^{(h)}}, \hat{\nu}^{(h)}+n_i^{\mathrm{o}}+2; \mathbb{A}_i^{\mathrm{c}})$ with $\hat{\boldsymbol{\mu}}_i^{\mathrm{c}\cdot\mathrm{o}^{(h)}}$ and $\hat{\boldsymbol{S}}_i^{\mathrm{cc}\cdot\mathrm{o}^{(h)}}$ being $\boldsymbol{\mu}_i^{\mathrm{c}\cdot\mathrm{o}}$ and $\boldsymbol{S}_i^{\mathrm{cc}\cdot\mathrm{o}}$, respectively, evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}^{(h)}$, and $\mathbb{A}_i^{\mathrm{c}}$ representing the left truncated region at $[\boldsymbol{w}_i^{\mathrm{c}} \leq \boldsymbol{u}_i^{\mathrm{c}}]$.

The desired quantities have been formulated. □

Web Appendix C: Proof of the missing information matrix of β.

The missing information matrix of $\boldsymbol{\beta}$ can be calculated as the conditional variance-covariance matrix of complete-data score vector with respect to $\boldsymbol{\beta}$, that is,

$$
\begin{aligned}
\boldsymbol{I}_m(\boldsymbol{\beta};\boldsymbol{y}) &= \sum_{i=1}^{N} \mathrm{Var}\Big(\frac{\partial \ell_c(\boldsymbol{\theta}|\mathcal{D}ata)}{\partial \boldsymbol{\beta}}\big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big) \\
&= \sum_{i=1}^{N} \mathrm{Var}\Big(\tau_i \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big) \\
&= \sum_{i=1}^{N} \Big\{\mathrm{Var}_{\boldsymbol{y}_i}\Big(E_{\tau_i}\Big[\tau_i \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\big|\boldsymbol{y}_i\Big]\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big) \\
&\quad + E_{\boldsymbol{y}_i}\Big(\mathrm{Var}_{\tau_i}\Big[\tau_i \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\big|\boldsymbol{y}_i\Big]\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big)\Big\} \\
&= \sum_{i=1}^{N} \Big\{\mathrm{Var}_{\boldsymbol{y}_i}\Big(\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big) \\
&\quad + E_{\boldsymbol{y}_i}\Big(\Big[\frac{2(\nu+n_i)}{(\nu+\Delta_{\boldsymbol{y}_i})^2}\Big]\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big)\Big\} \\
&= \sum_{i=1}^{N} \Big\{E_{\boldsymbol{y}_i}\Big(\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big) \\
&\quad - E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big]E^{\mathrm{T}}_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] \\
&\quad + E_{\boldsymbol{y}_i}\Big[\Big(\frac{2}{\nu+n_i}\Big)\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big]\Big\} \\
&= \sum_{i=1}^{N} \Big\{\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}E_{\boldsymbol{y}_i}\Big(\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big)\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i \\
&\quad - \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big]E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big]\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i \\
&\quad + \Big(\frac{2}{\nu+n_i}\Big)\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}E_{\boldsymbol{y}_i}\Big(\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})(\boldsymbol{y}_i-\boldsymbol{X}_i\boldsymbol{\beta})^{\mathrm{T}}\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big)\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i\Big\} \\
&= \sum_{i=1}^{N} \Big\{\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\Lambda}_i^{-1}\Big[\Big(1+\frac{2}{\nu+n_i}\Big)\Big(\widehat{\tau^2\boldsymbol{y}_i^2}-\widehat{\tau^2\boldsymbol{y}_i}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{X}_i^{\mathrm{T}}-\boldsymbol{X}_i\boldsymbol{\beta}\widehat{\tau^2\boldsymbol{y}_i}^{\mathrm{T}}+\widehat{\tau_i^2}\boldsymbol{X}_i\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{X}_i^{\mathrm{T}}\Big) \\
&\quad - \big(\widehat{\tau\boldsymbol{y}}_i-\hat{\tau}_i\boldsymbol{X}_i\boldsymbol{\beta}\big)\big(\widehat{\tau\boldsymbol{y}}_i-\hat{\tau}_i\boldsymbol{X}_i\boldsymbol{\beta}\big)^{\mathrm{T}}\Big]\boldsymbol{\Lambda}_i^{-1}\boldsymbol{X}_i,
\end{aligned}
$$

where $\widehat{\tau\boldsymbol{y}}_i$ and $\hat{\tau}_i$ are $\widehat{\tau\boldsymbol{y}}_i^{(h)}$ and $\hat{\tau}_i^{(h)}$ given in Propositions 2 and 3, respectively, with $\hat{\boldsymbol{\theta}}^{(h)}$ replaced by $\boldsymbol{\theta}$; and the proof of expressions for $\widehat{\tau_i^2}^{(h)}$, $\widehat{\tau^2\boldsymbol{y}_i}^{(h)}$, and $\widehat{\tau^2\boldsymbol{y}_i^2}^{(h)}$ are derived as follows under three censoring situations:

1. If the ith subject has only non-censored measurements, then it is straightforward that

$$\widehat{\tau_i^2} = \Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2, \quad \widehat{\tau^2\boldsymbol{y}_i} = \Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i, \text{ and } \widehat{\tau^2\boldsymbol{y}_i^2} = \Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}.$$

2. If the ith subject has only censored measurements, then we use the property of Proposition 2 of Matos et al. (2013) to derive the following conditional expectation:

$$\begin{aligned}\widehat{\tau_i^2} &= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] = c_{n_i}(\nu,2)\frac{\mathcal{T}_{n_i}(\boldsymbol{u}_i|\boldsymbol{X}_i\boldsymbol{\beta},\frac{\nu}{\nu+4}\boldsymbol{\Lambda}_i,\nu+4)}{\mathcal{T}_{n_i}(\boldsymbol{u}_i|\boldsymbol{X}_i\boldsymbol{\beta},\boldsymbol{\Lambda}_i,\nu)},\\ \widehat{\tau^2\boldsymbol{y}_i} &= E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] = \widehat{\tau_i^2}E[\boldsymbol{W}_i^*],\end{aligned}$$

and

$$\widehat{\tau^2\boldsymbol{y}^2} = E_{\boldsymbol{y}_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] = \widehat{\tau_i^2}E(\boldsymbol{W}_i^*\boldsymbol{W}_i^{*\mathrm{T}}),$$

where $\boldsymbol{W}_i^* \sim \mathcal{T}t_{n_i}(\boldsymbol{X}_i\boldsymbol{\beta},\frac{\nu}{\nu+4}\boldsymbol{\Lambda}_i,\nu+4,\mathbb{A}_i)$, and

$$c_{n_i}(\nu,2) = \big(\frac{\nu+n_i}{\nu}\big)^2\frac{\Gamma\big(\frac{n_i+\nu}{2}\big)\Gamma\big(\frac{\nu+4}{2}\big)}{\Gamma\big(\frac{\nu}{2}\big)\Gamma\big(\frac{n_i+\nu+4}{2}\big)} = \frac{(\nu+n_i)(\nu+2)}{\nu(n_i+\nu+2)}.$$

3. If the ith subject has both censored and non-censored measurements, then

$$\begin{aligned}\widehat{\tau_i^2} &= E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] = E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big\{E_{\tau_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2|\boldsymbol{y}_i\Big]\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big\}\\ &= \frac{d_{n_i}(n_i^{\mathrm{o}},\nu,2)}{(\nu+\Delta_{\boldsymbol{y}_i^{\mathrm{o}}})^2}\frac{\mathcal{T}_{n_i-n_i^{\mathrm{o}}}(\boldsymbol{u}_i^{\mathrm{c}}|\boldsymbol{\mu}_i^{\mathrm{c\cdot o}},\frac{\nu+n_i^{\mathrm{o}}}{\nu+n_i^{\mathrm{o}}+4}\boldsymbol{S}_i^{\mathrm{cc\cdot o}},\nu+n_i^{\mathrm{o}}+4)}{\mathcal{T}_{n_i-n_i^{\mathrm{o}}}(\boldsymbol{u}_i^{\mathrm{c}}|\boldsymbol{\mu}_i^{\mathrm{c\cdot o}}\boldsymbol{S}_i^{\mathrm{cc\cdot o}},\nu+n_i^{\mathrm{o}})},\\ \widehat{\tau^2\boldsymbol{y}_i} &= E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] = E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big\{E_{\tau_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i|\boldsymbol{y}_i\Big]\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big\}\\ &= \widehat{\tau_i^2}E[\boldsymbol{W}_i^{\mathrm{c}*}],\end{aligned}$$

$$\begin{aligned}\widehat{\tau^2\boldsymbol{y}^2}_i &= E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big] = E_{\boldsymbol{y}_i^{\mathrm{c}}}\Big\{E_{\tau_i}\Big[\Big(\frac{\nu+n_i}{\nu+\Delta_{\boldsymbol{y}_i}}\Big)^2\boldsymbol{y}_i\boldsymbol{y}_i^{\mathrm{T}}\Big|\boldsymbol{y}_i\Big]\Big|\boldsymbol{u}_i,\boldsymbol{c}_i\Big\}\\ &= \widehat{\tau_i^2}E[\boldsymbol{W}_i^{\mathrm{c}*}\boldsymbol{W}_i^{\mathrm{c}*^{\mathrm{T}}}],\end{aligned}$$

where $\boldsymbol{W}_i^{\mathrm{c}*} \sim \mathcal{T}t_{n_i-n_i^{\mathrm{o}}}(\boldsymbol{\mu}_i^{\mathrm{c\cdot o}},\frac{\nu+n_i^{\mathrm{o}}}{\nu+n_i^{\mathrm{o}}+4}\boldsymbol{S}_i^{\mathrm{cc\cdot o}},\nu+n_i^{\mathrm{o}}+4)$, and

$$d_{n_i}(n_i^{\mathrm{o}},\nu,2) = (\nu+n_i)^2\frac{\Gamma\big(\frac{n_i+\nu}{2}\big)\Gamma\big(\frac{n_i^{\mathrm{o}}+\nu+4}{2}\big)}{\Gamma\big(\frac{n_i^{\mathrm{o}}+\nu}{2}\big)\Gamma\big(\frac{n_i+\nu+4}{2}\big)} = \frac{(n_i+\nu)(n_i^{\mathrm{o}}+\nu+2)(n_i^{\mathrm{o}}+\nu)}{(n_i+\nu+2)}.$$

□

Web Appendix D: Supplementary table for the simulation study.

Table S.1. Simulation results for comparisons of four considered models in terms of model selection and parameter estimation under the scenario of $d = 0.50$ and $\varrho = 0.75$ based on 100 replications.

DOF	Censoring	Model	Model Fitting				Estimation of Fixed Effects							
			AIC	(Freq)	BIC	(Freq)	β_{10}	(SE)	β_{11}	(SE)	β_{20}	(SE)	β_{21}	(SE)
$\nu = 5$	5%	Nc	1714.13	(0)	1726.32	(0)	1.259	0.080	2.005	0.001	-1.716	0.061	4.003	5×10^{-4}
		Tc	1620.86	(100)	1634.27	(100)	1.063	0.081	2.002	0.001	-1.879	0.060	3.997	5×10^{-4}
		N	1726.48	(0)	1738.67	(0)	1.282	0.330	1.968	0.038	-1.677	0.281	3.961	0.028
		T	1651.39	(0)	1663.58	(0)	1.197	0.288	1.975	0.032	-1.716	0.252	3.968	0.023
	10%	Nc	1628.79	(0)	1640.98	(1)	1.204	0.073	2.015	0.001	-1.747	0.058	4.010	4×10^{-4}
		Tc	1541.42	(100)	1554.83	(99)	1.016	0.071	2.011	0.001	-1.858	0.056	3.998	4×10^{-4}
		N	1759.11	(0)	1771.30	(0)	1.798	0.316	1.899	0.038	-1.071	0.279	3.878	0.030
		T	1698.21	(0)	1710.40	(0)	1.713	0.289	1.909	0.032	-1.053	0.263	3.884	0.026
	20%	Nc	1491.82	(0)	1504.01	(0)	1.215	0.061	2.011	0.001	-1.793	0.055	4.016	4×10^{-4}
		Tc	1405.51	(100)	1418.92	(100)	1.068	0.062	1.997	0.001	-1.918	0.051	3.998	3×10^{-4}
		N	1843.98	(0)	1856.17	(0)	3.262	0.301	1.708	0.039	0.156	0.292	3.713	0.037
		T	1806.64	(0)	1818.83	(0)	3.208	0.302	1.714	0.036	0.170	0.298	3.718	0.035
$\nu = 50$	5%	N	1486.05	(88)	1498.24	(93)	0.952	0.077	2.005	0.001	-1.993	0.056	3.997	5×10^{-4}
		Tc	1487.57	(12)	1500.98	(7)	0.954	0.077	2.005	0.001	-1.992	0.055	3.997	5×10^{-4}
		N	1516.06	(0)	1528.25	(0)	1.146	0.271	1.979	0.031	-1.770	0.230	3.967	0.022
		T	1514.78	(0)	1526.97	(0)	1.142	0.277	1.979	0.031	-1.766	0.243	3.967	0.022
	10%	Nc	1407.98	(89)	1420.17	(97)	0.868	0.070	2.017	0.001	-2.014	0.054	4.001	4×10^{-4}
		Tc	1409.52	(11)	1422.93	(3)	0.873	0.071	2.016	0.001	-2.012	0.053	4.000	4×10^{-4}
		N	1572.64	(0)	1584.83	(0)	1.650	0.259	1.911	0.031	-1.164	0.231	3.884	0.025
		T	1571.75	(0)	1583.93	(0)	1.652	0.277	1.912	0.030	-1.147	0.254	3.885	0.025
	20%	Nc	1275.91	(93)	1288.10	(98)	0.987	0.058	2.000	0.001	-2.001	0.050	3.999	4×10^{-4}
		Tc	1277.56	(7)	1290.96	(2)	0.989	0.063	2.000	0.001	-2.002	0.050	3.999	3×10^{-4}
		N	1694.51	(0)	1706.70	(0)	3.133	0.252	1.716	0.033	0.090	0.251	3.716	0.033
		T	1694.33	(0)	1706.52	(0)	3.137	0.288	1.717	0.033	0.101	0.287	3.716	0.033

Nc: MLMMC; Tc: MtLMMC; N: MLMM; T: MtLMM